Quiz 4: Solutions

Problem 1. Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.

(a) $f(n) = n - 1$; (b) $f(n) = n^2 + 1$; (c) $f(n) = n^3$; (d) $f(n) = \lceil n/2 \rceil$.

Solution. (a) $f$ is one-to-one because $f$ is increasing; (b) $f$ is not one-to-one, e.g., $f(1) = f(-1) = 2$; (c) $f$ is one-to-one because $f$ is increasing; (d) $f$ is not one-to-one, e.g., $f(1) = f(2) = 1$.

Problem 2. Find the value of each of these sums.

(a) $\sum_{j=0}^{8} (1 + (-1)^j)$

(b) $\sum_{j=0}^{8} (2^j - 2^j)$

(c) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j)$

(d) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

Hint: You may use the formula for the sum of a geometric progression:
$$\sum_{j=0}^{n} q^j = \frac{1 - q^{n+1}}{1 - q}, \quad q \neq 0, \neq 1.$$

Solution. (a) $\sum_{j=0}^{8} (1 + (-1)^j) = \sum_{k=0}^{4} (1 + (-1)^{2k}) = \sum_{k=0}^{4} 2 = 2 \cdot 5 = 10$,
where we used the identity $1 + (-1)^j = 0$ for odd $j$.

(b) $\sum_{j=0}^{8} (2^j - 2^j) = \sum_{j=0}^{8} 0 = 0$.

(c) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j) = 2 \cdot \frac{1 - 3^9}{1 - 3} + 3 \cdot \frac{1 - 2^9}{1 - 2} = 3^9 - 1 + 3 \cdot (2^9 - 1) = 3^9 + 3 \cdot 2^9 - 4 = 21215$.

(d) $\sum_{j=0}^{8} (2^{j+1} - 2^j) = \sum_{j=0}^{8} (2 \cdot 2^j - 2^j) = \sum_{j=0}^{8} 2^j = \frac{1 - 2^9}{1 - 2} = 2^9 - 1 = 511$. 