Exam 2: Solutions

MATH 221 (Section 002)

Problem 1. Given matrices
\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \]
find (a) \( A \odot B \); (b) \( B \odot A \).

Solution. (a)
\[
A \odot B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1 \land 1) \lor (1 \land 0) \lor (0 \land 1) & (1 \land 1) \lor (1 \land 1) \lor (0 \land 0) \\ (0 \land 1) \lor (1 \land 0) \lor (1 \land 1) & (0 \land 1) \lor (1 \land 1) \lor (1 \land 0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

(b)
\[
B \odot A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} (1 \land 1) \lor (1 \land 0) & (1 \land 1) \lor (1 \land 1) & (1 \land 0) \lor (1 \land 1) \\ (0 \land 1) \lor (1 \land 0) & (0 \land 1) \lor (1 \land 1) & (0 \land 0) \lor (1 \land 1) \\ (1 \land 1) \lor (0 \land 0) & (1 \land 1) \lor (0 \land 1) & (1 \land 0) \lor (0 \land 1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]

Problem 2. Use a proof by cases to show that \( \lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor \) for all integer \( n \).

Solution. Case 1: \( n = 2k \) for some integer \( k \). Then \( \lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor k \rfloor \lceil k \rceil = k^2 \), and \( \lfloor n^2/4 \rfloor = \lceil k^2 \rceil = k^2 \).

Case 2: \( n = 2k + 1 \) for some integer \( k \). Then \( \lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor k + 1/2 \rfloor \lceil k + 1/2 \rceil = k(k + 1) = k^2 + k \), and \( \lfloor n^2/4 \rfloor = \lceil k^2 + k + 1/4 \rceil = k^2 + k \).

Problem 3. Prove that if \( m \) and \( n \) are positive integers and \( x \) is a real number, then
\[
\left\lfloor \frac{x}{m} + \frac{n}{m} \right\rfloor = \left\lfloor \frac{x + n}{m} \right\rfloor.
\]
Solution. Let \(|x| + n = mq + r\) where \(q, r \in \mathbb{N}, r < m\). Then we have

\[
\left\lfloor \frac{|x| + n}{m} \right\rfloor = \left\lfloor \frac{mq + r}{m} \right\rfloor = \left\lfloor \frac{q + \frac{r}{m}}{m} \right\rfloor = q
\]

\[
\leq \left\lfloor \frac{x + n}{m} \right\rfloor \leq \frac{x + n}{m} < \left\lfloor \frac{|x| + 1 + n}{m} \right\rfloor = \frac{mq + r + 1}{m} = q + \frac{r + 1}{m} \leq q + 1,
\]

where we used that \(r + 1 \leq m\). Since the inequalities above include

\[
q \leq \left\lfloor \frac{x + n}{m} \right\rfloor < q + 1,
\]

and \(\left\lfloor \frac{x + n}{m} \right\rfloor\) is an integer, it must be equal to \(q\).

**Problem 4.** Use mathematical induction to prove that \(n! < n^n\) whenever \(n\) is a positive integer greater than 1.

**Solution.**

(1) Induction base. For \(n = 2\) we have \(2! = 2 < 4 = 2^2\).

(2) Induction step. Suppose the statement is true for \(n = k\), that is, \(k! < k^k\). Then

\[
(k + 1)! = k!(k + 1) < k^k(k + 1) < (k + 1)^k(k + 1) = (k + 1)^{k+1},
\]

i.e., the statement is true for \(n = k + 1\).

We conclude that the statement is true for all positive integers greater than 1.

**Problem 5.** Give a recursive definition of the sequence \(\{a_n\}, n = 1, 2, \ldots\) if (a) \(a_n = 4n - 2\); (b) \(a_n = 1 + (-1)^n\); (c) \(a_n = n(n + 1)\); (d) \(a_n = n^2\).

**Solution.** There are many equivalent ways to give a recursive definition of the sequence, so if my answers here are different from yours, it does not necessarily mean that your answers are incorrect.

(a) \(a_1 = 2, a_{n+1} = a_n + 4\).
(b) \(a_1 = 0, a_{n+1} = a_n + 2(-1)^{n+1}\).
(c) \(a_1 = 2, a_{n+1} = a_n \frac{n+2}{n}\).
(d) \(a_1 = 1, a_{n+1} = a_n + 2n + 1\).

**Problem 6.** Let

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Show that

\[
A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}
\]

whenever \(n\) is an integer greater than 1, where \(f_n, n = 1, 2, \ldots\), are the Fibonacci numbers.

**Hint.** Use mathematical induction.
Solution. (1) Induction base. For \( n = 1 \) we have
\[
A^1 = A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix}.
\]

(2) Induction step. Suppose the statement is true for \( n = k \), that is,
\[
A^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}.
\]

Then
\[
A^{k+1} = A^k A = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{bmatrix} = \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix},
\]
i.e., the statement is true for \( n = k + 1 \).

We conclude that the statement is true for all positive integer \( n \).

**Problem 7.** How many strings of four decimal digits
(a) do not contain the same digit twice?
(b) end with an even digit?
(c) have exactly three digits that are 9s?

**Solution.** (a) \( P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040 \).
(b) \( 10 \cdot 10 \cdot 10 \cdot 5 = 5000 \).
(c) The remaining digit must be different from 9 and can be at any of the 4 positions. Then there are \( 9 \cdot 4 = 36 \) such strings.

**Problem 8.** There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

**Solution.** At least \( \lceil 677/38 \rceil = 18 \) rooms.

**Problem 9.** A club has 25 members.
(a) How many ways are there to choose four members of the club to serve as an executive committee?
(b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club?

**Solution.** (a) \( C(25, 4) = \frac{25!}{4!21!} = \frac{25 \cdot 24 \cdot 23 \cdot 22}{1 \cdot 2 \cdot 3 \cdot 4} = 12650 \).
(b) \( P(25, 4) = 25 \cdot 24 \cdot 23 \cdot 22 = 303600 \).

**Problem 10.** What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

**Solution.** Let \( A \) be the set of such integers divisible by 5, and let \( B \) be the set of such integers divisible by 7. We have to find the probability of the event \( A \cup B \). The computation is using the inclusion-exclusion formula.
\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{|A| + |B| - |A \cap B|}{100} = \frac{20 + 14 - 2}{100} = \frac{32}{100} = 0.32
\]
Problem 11 (extra credit). Find the solution to each of the following recurrence relations and initial conditions.

(a) \( a_n = 3a_{n-1}, \ a_0 = 2. \)
(b) \( a_n = a_{n-1} + 2, \ a_0 = 3. \)
(c) \( a_n = a_{n-1} + n, \ a_0 = 1. \)

Solution. (a) \( a_n = 2 \cdot 3^n. \)
(b) \( a_n = 3 + 2n. \)
(c) \( a_n = 1 + (1 + 2 + \cdots + n) = 1 + \frac{n(n+1)}{2} = \frac{n^2 + n + 2}{2}. \)