Sample Final Exam

Problem 1. (a) Show that the statements \( \neg (p \oplus q) \) and \( p \leftrightarrow q \) are logically equivalent.

(b) Show that \([ (p \lor q) \land (p \rightarrow r) \land (q \rightarrow r) ] \rightarrow r \) is a tautology.
Problem 2. Determine the truth value of each of the following statements if the universe of discourse of each variable is the set of real numbers.

(a) $\forall x \exists y (x^2 = y)$;
(b) $\forall x \exists y (x = y^2)$;
(c) $\forall x \neq 0 \exists y (xy = 1)$;
(d) $\exists x \forall y \neq 0 (xy = 1)$. 
Problem 3. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of the following sets with bit strings where the $i$th bit in the string is 1 if $i$ is in the set and 0 otherwise.

(a) $\{3, 4, 5\}$;
(b) $\{1, 3, 6, 10\}$;
(c) $\{2, 3, 4, 7, 8, 9\}$.
Problem 4. Using the same universal set as in Problem 3, find the set specified by each of the following strings.

(a) 1111001111;
(b) 010111000;
(c) 1000000001.
**Problem 5.** Give a big-O estimate for each of the following functions. For the function $g$ in your estimate that $f(n)$ is $O(g(n))$ use a function $g$ of the form $g(n) = n^k$ with the smallest possible $k$.

(a) $f(n) = n \log(n^2 + 1) + n^2 \log n$;
(b) $f(n) = (n \log n + 1)^2 + (\log n + 1)(n^2 + 1)$. 
Problem 6. What is the greatest common divisor and the least common multiple of the following pair of integers:

$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$?
Problem 7. Prove:

(a) that if $n$ is a positive integer, then $n$ is odd if and only if $5n + 6$ is odd.

(b) that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of prove did you use?
Problem 8. Use mathematical induction to prove that
\[ \sum_{k=1}^{n} k \cdot 2^k = (n - 1)2^{n+1} + 2. \]
Problem 9. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

(a) Show that there are at least 9 freshman, at least 9 sophomores, or at least 9 juniors in the class.

(b) Show that there are either at least 3 freshman, at least 19 sophomores, or at least 5 juniors in the class.
Problem 10. Thirteen people on a softball team show up for a game.

(a) How many ways are there to choose 10 players to take the field?

(b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

(c) Of the 13 people who show up, 3 are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?