WA 5: Solutions

Problem 1. Find the general form of a linear fractional transformation of the upper half plane \( \text{Im} \ z > 0 \) onto itself.

Solution. Since a linear transformation which maps the upper half plane onto itself, maps its boundary, the real line, into itself and this linear transformation is a bijection of \( \mathbb{C} \) onto itself, there exist three distinct real numbers \( z_1, z_2, z_2 \) whose images \( w_1, w_2, w_3 \) are also distinct real numbers. The transformation is uniquely determined by the equation

\[
\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.
\]

Solving for \( w \), we obtain that

\[
w = \frac{az + b}{cz + d},
\]

with \( a, b, c, d \in \mathbb{R} \). This transformation must also satisfy \( \text{Im} \ w(i) > 0 \), which is equivalent to

\[
\text{Im} \left( \frac{ai + b}{ci + d} \right) = \frac{ad - bc}{c^2 + d^2} > 0.
\]

Since \( ad - bc \neq 0 \), we have \( c^2 + d^2 > 0 \), and thus \( ad - bc > 0 \). Therefore, the general form of a linear fractional transformation of the upper half plane \( \text{Im} \ z > 0 \) onto itself is

\[
w = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}: \ ad - bc > 0.
\]

Problem 2. A fixed point of a transformation \( w = f(z) \) is a point \( z_0 \) such that \( f(z_0) = z_0 \). Show that every linear fractional transformation, with the exception of the identity transformation, has at most two fixed points in the extended plane.

Solution. Let the LFT be given by \( w = \frac{az + b}{cz + d} \) with some \( a, b, c, d \in \mathbb{C} \) such that \( ad - bc \neq 0 \). Consider the following two cases:

Case 1: \( c = 0 \). In this case, \( a \neq 0, d \neq 0 \) and \( w = \frac{a}{d}z + \frac{b}{d} \). Clearly, \( w(\infty) = \infty \), thus \( \infty \) is a fixed point. Any other fixed point must satisfy \( z = \frac{a}{d}z + \frac{b}{d} \), which has a single solution unless \( a = d \) and \( b = 0 \), i.e., in the case of the identity transformation.

Case 2: \( c \neq 0 \). In this case, \( w(\infty) = \frac{a}{c} \neq \infty \), i.e., \( \infty \) is not a fixed point. The equation for a fixed point, \( z = \frac{az + b}{cz + d} \) is equivalent to \( az + b = cz^2 + dz \), which is quadratic, and therefore has at most two solutions.

Problem 3. Show that the cross-ratio \( \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \) of the four distinct complex points \( z_1, z_2, z_3, z_4 \) is real if and only if all the four points lie on a circle or on a straight line.

Solution. Define the following linear fractional transformation:

\[
w(z) = \frac{(z - z_2)(z_3 - z_4)}{(z - z_4)(z_3 - z_2)}.
\]

Obviously, \( w(z_2) = 0 \), \( w(z_3) = 1 \), \( w(z_4) = \infty \). Thus this transformation maps a circle or a straight line (which is uniquely determined) through the points \( z_2, z_3, z_4 \) onto the real line. Then the given cross-ratio, which is equal to \( w(z_1) \) is real if and
only if the forth point $z_1$ lies on the same circle or the same straight line as the points $z_2, z_3, z_4$.

**Problem 4.** Find a linear-fractional transformation that maps distinct points $z_1, z_2, z_3$ of the complex plane onto the points $w_1 = 0, w_2 = 1, w_3 = \infty$, respectively.

**Solution.**

$$w(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

**Problem 5.** Find a linear transformation $T(\cdot)$ such that the function $w = T(z^2)^{1/2}$, with the principal branch of the square root chosen, maps 0 to 0 and the hyperbola $xy = 1$ onto the hyperbola $u^2 - v^2 = 1$.

**Solution.** Let $T(z) = Az + B$ be the desired linear transformation. Since $T(0) = 0$, we must have $B = 0$. Next, let $z = x + iy$ satisfy $xy = 1$ and $w = u + iv$ satisfy $u^2 - v^2 = 1$. Then $z^2 = x^2 - y^2 + 2xyi$ can be any point on the line $\text{Im } z = 2$, and the corresponding $w^2 = u^2 - v^2 + 2uvi$ is a point on the line $\text{Re } w = 1$. Since we have $w^2 = T(z^2) = Az^2$, we must have $\text{Re } A(a + 2i) = 1$ for every $a \in \mathbb{R}$. Hence, $a\text{Re } A - 2\text{Im } A = 1$ for all $a \in \mathbb{R}$. This is possible only for $\text{Im } A = -\frac{1}{2}$, $\text{Re } A = 0$, i.e., when $A = -\frac{i}{2}$. We conclude that $T(z) = -\frac{i}{2}z$. 