Problem 1. Suppose \( \Gamma \) is an analytic function in \( \mathbb{C} \) except for a set of nonpositive integers, where it has simple poles. Suppose, moreover, that \( \Gamma(1) = 1 \) and \( z \Gamma(z) = \Gamma(z + 1) \) for all regular points. Prove that
\[
\text{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}, \quad n = 0, 1, \ldots.
\]

Solution. We have
\[
\Gamma(z) = \Gamma(z + 1) = \Gamma(z + 2) \cdot \frac{\Gamma(z + 1)}{z(z + 1)} = \cdots = \frac{\Gamma(z + n + 1)}{z(z + 1) \cdots (z + n - 1)(z + n)}.
\]
Since \( \Gamma(z) \) is analytic at \( z = 1 \), we conclude that \( \Gamma(z) \) has a simple pole at \( z = -n \) for every \( n = 0, 1, 2, \ldots \), and one has for \( n = 0 \) that
\[
\text{Res}_{z=0} \Gamma(z) = \Gamma(1) = 1 = \frac{(-1)^0}{0!},
\]
and for \( n = 1, 2, \ldots \) that
\[
\text{Res}_{z=-n} \Gamma(z) = \Gamma(1) = 1 = \frac{(-1)^n}{n!}.
\]

Problem 2. If \( z_\nu \) is a pole of \( \frac{1}{z^4 + a^4} \), show that
\[
\text{Res}_{z=z_\nu} \frac{1}{z^4 + a^4} = -\frac{z_\nu}{4a^4}.
\]

Solution. Since \( g(z) = z^4 + a^4 \) has 4 simple zeros at the 4-th roots of \(-a^4\), the function \( f(z) = \frac{1}{g(z)} \) has 4 simple poles at those points. If \( z_\nu \) is one of these poles then
\[
\text{Res}_{z=z_\nu} \frac{1}{z^4 + a^4} = \frac{1}{(z^4 + a^4)'|_{z=z_\nu}} = \frac{1}{4z_\nu^3} = \frac{z_\nu}{4a^4} = -\frac{z_\nu}{4a^4}.
\]

Problem 3. Find \( \int_C \frac{dz}{1+z^3} \) where \( C \) is the positively oriented ellipse \( 2x^2 + y^2 = \frac{3}{2} \).

Solution. The function \( f(z) = \frac{1}{1+z^3} \) has simple poles at \( z_1 = -1 \), \( z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \), and \( z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i \), of which only \( z_2 \) and \( z_3 \) are inside the ellipse \( 2x^2 + y^2 = \frac{3}{2} \). Therefore, by the Cauchy residue theorem
\[
\int_C \frac{dz}{1+z^3} = 2\pi i \left( \text{Res}_{z=z_2} \frac{1}{1+z^3} + \text{Res}_{z=z_3} \frac{1}{1+z^3} \right) = 2\pi i \left( \frac{1}{3z_2^2} + \frac{1}{3z_3^2} \right) = -\frac{2\pi i}{3}.
\]

Problem 4. In each case, write the Laurent series of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point, and find the residue at that point:
(a) \( ze^{1/z} \); (b) \( \frac{z^2}{1+z^2} \); (c) \( \frac{1-e^{z^3}}{z^3} \); (d) \( (2-z)^{-3} \).
Solution. (a) The function \( f(z) = z e^{1/z} \) has the only singular point at \( z = 0 \) and its Laurent series is given by
\[
ze^{1/z} = z \left( 1 + \frac{1}{1!} z + \frac{1}{2!} \frac{1}{z} + \cdots + \frac{1}{n!} z^n + \cdots \right) = z + 1 + \frac{1}{2z} + \frac{1}{6z^2} + \cdots.
\]
Thus \( z = 0 \) is an essential singular point and \( \text{Res}_{z=0} \left( z e^{1/z} \right) = \frac{1}{2} \).

(b) The function \( f(z) = \frac{z^2}{1+z^2} \) has the only singular point at \( z = -1 \) and its Laurent series is given by
\[
\frac{z^2}{1+z} = \frac{z^2 - 1 + 1}{1+z} = z - 1 + \frac{1}{1+z} = (z+1) - 2 + \frac{1}{z+1}.
\]
Thus \( z = -1 \) is a simple pole and \( \text{Res}_{z=-1} \left( \frac{z^2}{1+z^2} \right) = 1 \).

(c) The function \( f(z) = \frac{1 - \cosh z}{z^3} \) has the only singular point at \( z = 0 \) and its Laurent series is given by
\[
\frac{1 - \cosh z}{z^3} = \frac{1}{z^3} \left( 1 - 1 - \frac{z^2}{2!} - \frac{z^4}{4!} - \cdots + \frac{z^{2n}}{(2n)!} + \cdots \right) = -\frac{1}{2z} - \frac{z}{24} - \cdots.
\]
Thus \( z = 0 \) is a simple pole and \( \text{Res}_{z=0} \left( \frac{1 - \cosh z}{z^3} \right) = -\frac{1}{2} \).

(d) The function \( f(z) = (2 - z)^{-3} \) has the only singular point at \( z = 2 \) and its Laurent series is given by
\[
(2 - z)^{-3} = -(z - 2)^{-3}.
\]
Thus \( z = 2 \) is a pole of order \( 3 \) and \( \text{Res}_{z=2} \left( (2 - z)^{-3} \right) = 0 \).

Problem 5. Use the theorem involving a single residue to evaluate the integral of \( f(z) \) around the positively oriented circle \( |z| = 3 \) when
(a) \( f(z) = \frac{(3z + 2)^2}{z(z-1)(2z+5)} \); (b) \( f(z) = \frac{z^3(1-3z)^2}{(1+z)(1+z+2z^2)} \); (c) \( f(z) = \frac{z^3 e^{1/z}}{1+z} \).

Solution. (a) The function \( f(z) = \frac{(3z + 2)^2}{z(z-1)(2z+5)} \) has simple poles at \( z = 0, z = 1, \) and \( z = -5/2, \) all of which lie inside the circle \( C_3 = \{ z \in \mathbb{C} : |z| = 3 \} \). Therefore,
\[
\int_{C_3} \frac{(3z + 2)^2}{z(z-1)(2z+5)} \, dz = 2\pi i \text{Res}_{z=0} \left( \frac{(3z + 2)^2}{z(z-1)(2z+5)} \right) = 2\pi i \left( \frac{2}{5} \right) = \frac{9\pi i}{2}.
\]

(b) The function \( f(z) = \frac{z^3(1-3z)^2}{(1+z)(1+z+2z^2)} \) has simple poles at \( z = -1, \) and at 4 roots of order 4 of \(-1/2, \) all of which lie inside \( C_3 \). Therefore,
\[
\int_{C_3} \frac{z^3(1-3z)}{(1+z)(1+2z^2)} \, dz = 2\pi i \text{Res}_{z=-1} \left( \frac{z^3(1-3z)}{(1+z)(1+2z^2)} \right) = 2\pi i \left( \frac{3}{2} \right) = -3\pi i.
\]
(c) The function $f(z) = \frac{z^3 e^{1/z}}{1 + z^3}$ has simple poles at the 3 roots of order 3 of $-1$ and an essential singular point at $z = 0$, and all of these singular points lie inside $C_3$. Therefore,

$$\int_{C_3} \frac{z^3 e^{1/z}}{1 + z^3} \, dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} \frac{1}{1 + 1/z^3}$$

$$= 2\pi i \operatorname{Res}_{z=0} \frac{e^z}{z^2(z^3 + 1)} = 2\pi i \cdot \frac{d}{dz} \left( \frac{e^z}{z^3 + 1} \right) \bigg|_{z=0} = 2\pi i.$$