Inverse Functions

Definition: Functions \( f \) and \( g \) are inverse functions if

\[ f(g(x)) = x \quad \text{for all} \quad x \quad \text{in the domain of} \quad f, \]

and

\[ g(f(x)) = x \quad \text{for all} \quad x \quad \text{in the domain of} \quad g. \]

Remark: The inverse of a function \( f \) is written \( f^{-1} \) (read "inverse of \( f \)").
\[ \cap \{ y = f(x) \} \quad \text{and} \quad y = f^{-1}(x) \]

Illustration:

\[ f \quad \cap \quad x \quad y \]
\[ f(x) = x^2 = 9 \]

\[ \text{domain } x \in [0, \infty) \]

\[ \text{ex} \quad g(\sqrt{2}) = 9 \]

\[ g(\sqrt{x}) = g(x^2) = x \quad \text{for all } f \]

\[ f(g(x)) = f(x^2) = \sqrt{x^2} = x \]

\[ \text{Check} \]

\[ \text{domain of } g(x) \quad (-\infty, \infty) \]

\[ \text{domain of } f(x) \quad [0, \infty) \]

\[ f(g(x)) = x^2 = x \]

\[ \text{Guess that } g(x) = \sqrt{x} \text{ is the inverse of } f(x). \]
The function $f^{-1}(x)$ maps to the values $a$ and $-a$.

Thus, $f^{-1}(x) \neq x^2$.

In fact, $f(x) = x^2$ does not have an inverse function.
H.L.T. does not pass the H.L.T.

\[ \text{If } f(x) = x^3 \text{ on } (-\infty, \infty), \text{ then } \exists f^{-1}(x) \text{ on } \text{pass the H.L.T.} \]

None of \( f(x) = x^3 \) on \((-\infty, \infty)\) pass the H.L.T.

\[ \text{If } f \text{ is one-to-one, } \text{it is called one-to-one.} \]

\( f \text{ is one-to-one if and only if any horizontal line intersects the curve of } f \text{ at most once.} \)

\[ \text{Horizontal line test (H.L.T.)} \]

A function \( f \) has an inverse if its graph passes the Horizontal line test.
Finding inverses

Recall

\[ x = \frac{-y}{y^2 + 5} \]

Step 1: Solve for x

\[ y = \frac{-x}{x^2 + 5} \]

Recall

Let \( f(x) = \sqrt{2x - 5} \) and find its domain.

Find a formula for \( f^{-1}(x) \) and find its domain.
The range of $f^{-1}$ is the domain of $f$.

The domain of $f^{-1}$ is the range of $f$.

Recall:

\[ f^{-1}(x) = \frac{x - \sqrt{2}}{s} \]

Step 2: Evaluate $f^{-1}$ at $x$. 

For all \( x \) in domain \( f \)

\[
\text{range } f^{-1} = \text{domain } f = \left[ \frac{\sqrt{2}}{2}, \infty \right)
\]

Thus domain \( f^{-1} \) = range \( f \) = \( \left[ 0, \infty \right) \)

Range of \( f \): \( y \geq 0 \)

\( \text{range of } f : 2x - 5 \leq 0 \leq x^2 \geq \frac{\sqrt{2}}{2} \)

\[
\alpha = \frac{2}{x^2 + 5}
\]

Previous ax: \( f(x) = \sqrt{2x - 5} \) and \( f^{-1}(x) = x^2 + 5 \)
A graph of \( f \) is the graph of \( f^{-1} \) reflected over the line \( y = x \).

**Fact:** If a function \( f \) has an inverse \( f^{-1} \), then the graphs of \( f(x) \) and \( f^{-1}(x) \) are related.