Law of Sines & Law of Cosines

I Law of Sines
Consider the following triangle

\[ \triangle ABC \]

- Side \( a \) is opposite angle \( A \)
- Side \( b \) is opposite angle \( B \)
- Side \( c \) is opposite angle \( C \)

We can express the area of this triangle in 3 different ways.

### Way 1:
When we use the red triangle, \( \sin B = \frac{h}{a} \) \( \Rightarrow \) \( h = a \sin B \)

\[ \text{Thus } \begin{array}{l}
\text{Area} = \frac{1}{2} (c)(a \sin B)
\end{array} \]

### Way 2:
When we use the red triangle, \( \sin A = \frac{h}{b} \) \( \Rightarrow \) \( h = b \sin A \)

\[ \text{Thus } \begin{array}{l}
\text{Area} = \frac{1}{2} (c)(b \sin A)
\end{array} \]

### Way 3: If we rotate the triangle and let
- Side \( a \) be the base

\[ \triangle ABC \]

Using the red triangle, we see \( h = b \sin C \) and \( \text{Area} = \frac{1}{2} (a)(b \sin C) \)

Since all equations represent the same area, they must be equal!
i.e. \( \frac{1}{2} bc \sin (A) = \frac{1}{2} (ac \sin (B) = \frac{1}{2} (ab \sin C) \)

Multiplying all of these terms by 2 and dividing everything by \( abc \), we get

\[
\frac{\sin (A)}{a} = \frac{\sin (B)}{b} = \frac{\sin (C)}{c}
\]

for the triangle

\[
\begin{array}{c}
a \quad C \\
B \quad A \\
\end{array}
\]

This is called the law of sines.

Find the lengths of all three sides of the following triangle:

\[
\begin{array}{c}
35^\circ \\
10 \\
36^\circ \\
A \\
35^\circ \\
36^\circ \\
B \\
C \\
\end{array}
\]

Solution: In this triangle,

let \( B = 36^\circ \)

\( C = 55^\circ \)

Then \( A = 180^\circ - 36^\circ - 55^\circ = 89^\circ \)

and the triangle looks like:

\[
\begin{array}{c}
55^\circ \\
10 \\
89^\circ \\
A \\
35^\circ \\
B \\
C \\
\end{array}
\]

We can now apply the law of sines to find the lengths of \( a \) and \( c \) with

\( b = 10 \).

\[
\frac{\sin A}{a} = \frac{\sin B}{b} \quad \Rightarrow \quad \frac{\sin (89^\circ)}{a} = \frac{\sin (36^\circ)}{10}
\]

\[
\Rightarrow \quad a = \frac{10 \sin (89^\circ)}{\sin (36^\circ)} \approx 17.01
\]

And,

\[
\frac{\sin C}{c} = \frac{\sin B}{b} \quad \Rightarrow \quad \frac{\sin (55^\circ)}{c} = \frac{\sin (36^\circ)}{10}
\]

\[
\Rightarrow \quad c = \frac{10 \sin (55^\circ)}{\sin (36^\circ)} \approx 13.9
\]
Laws of Cosines

Consider the following triangle

We draw a line perpendicular to \( b \), splitting side \( b \) into 2 pieces (not necessarily equal)

Notice that \( \sin C = \frac{h}{a} \) \( \therefore h = a \sin C \)

\[ \cos C = \frac{y}{a} \therefore y = a \cos C \]

And, since \( x + y = b \)

\[ x + a \cos (C) = b \therefore x = b - a \cos (C) \]

Thus, our triangle looks like:

Using Pythagorean theorem on the red triangle gives us:

\[
\begin{align*}
    c^2 &= (b-a \cos C)^2 + (a \sin C)^2 \\
    &= b^2 - 2ab \cos C + a^2 \cos^2 (C) + a^2 \sin^2 (C) \\
    &= b^2 - 2ab \cos (C) + a^2 \left[ \cos^2 C + \sin^2 C \right] \\
    &= b^2 + a^2 - 2ab \cos (C)
\end{align*}
\]

Since \( \sin^2 \theta + \cos^2 \theta = 1 \)

And, we have derived the Law of Cosines:

\[
    c^2 = a^2 + b^2 - 2ab \cos (C)
\]

We'll do an example with this after we discuss inverse trigonometric functions.