I. Pythagorean Identities

Recall: \( \sin^2 \theta + \cos^2 \theta = 1 \) for any angle \( \theta \).

If we divide both sides by \( \cos^2 \theta \), we get

\[
\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta}
\]

\[
\left( \frac{\sin \theta}{\cos \theta} \right)^2 + 1 = \left( \frac{1}{\cos \theta} \right)^2
\]

\[
\tan^2 \theta + 1 = \sec^2 \theta
\]

i.e.

\[
\tan^2 \theta + 1 = \sec^2 \theta
\]

Similarly, if we divide every term by \( \sin^2 \theta \), we would get

\[
1 + \cot^2 \theta = \csc^2 \theta
\]

II. Identities with negative angles

Recall: \( (\sin \theta, \cos \theta) \) are the endpoints of the radius of the unit circle that forms an angle \( \theta \) with the positive horizontal axis.

\[
\text{Diagram showing } (\cos \theta, \sin \theta) \text{ and } (\cos(-\theta), \sin(-\theta))
\]

Notice:

1. \( \cos(-\theta) \) and \( \cos(\theta) \) are on the same vertical line.

\[
\therefore \cos(-\theta) = \cos(\theta)
\]

2. \( \sin(\theta) \) and \( \sin(-\theta) \) are reflections over the x-axis.

\[
\therefore \sin(-\theta) = -\sin(\theta)
\]
(3) Once we know about \( \sin(-\theta) \) and \( \cos(-\theta) \) in terms of \( \sin \theta \) and \( \cos \theta \), we can represent the other trig. funs of negative angles in terms of positive angles.

\[
\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} \]

\[\therefore \tan(-\theta) = -\tan \theta\]

III Trigonometric identities w/ \( \frac{\pi}{2} \).

Suppose \( 0 < \theta < \frac{\pi}{2} \) and consider the following triangle.

Notice: the third angle is \( \frac{\pi}{2} - \theta \) because the sum of all 3 angles should be \( \pi \) radians (i.e., 180°).

\[
\sin \theta = \frac{b}{c} \quad \sin \left( \frac{\pi}{2} - \theta \right) = \frac{a}{c} \\
\cos \theta = \frac{a}{c} \quad \cos \left( \frac{\pi}{2} - \theta \right) = \frac{b}{c} \\
\tan \theta = \frac{b}{a} \quad \tan \left( \frac{\pi}{2} - \theta \right) = \frac{a}{b}
\]

Thus,
\[
\sin \left( \frac{\pi}{2} - \theta \right) = \cos \theta \\
\cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta \\
\tan \left( \frac{\pi}{2} - \theta \right) = \frac{1}{\tan \theta} = \cot \theta
\]

*We derived these formulas for \( 0 < \theta < \frac{\pi}{2} \); however, they actually hold for all \( \theta \).
### Sum and Difference Formulas

Consider:

\[ (\cos A, \sin A) \]

\[ (\cos (-B), \sin (-B)) = (\cos B, -\sin B) \]

We will find formulas for \( \sin (A+B) \) and \( \cos (A+B) \).

To do this, let us examine the triangle:

![Diagram of triangle with labels (cosA, sinA), A+B, C, (cosB, -sinB)](image)

Notice that the total angle between the two radii is \( A+B \).

We will compute the length \( \overline{BC} = \overline{C} \) in 2 different ways:

1. **By the distance formula**

   \[
   C = \sqrt{(\cos A - \cos B)^2 + (\sin A - (-\sin B))^2}
   \]

   \[
   C = \sqrt{(\cos A - \cos B)^2 + (\sin A + \sin B)^2}
   \]

   \[
   C^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2
   \]

   \[
   = \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B
   \]

   \[
   = (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2\cos A\cos B + 2\sin A\sin B
   \]

   \[
   C^2 = 2 - 2\cos A\cos B + 2\sin A\sin B
   \]

2. **We can also compute \( C^2 \) by the law of cosines.**

   \[ c^2 = a^2 + b^2 - 2ab \cos C \]

   Notice that \( a = 1 \), \( b = 1 \), and \( c = A+B \).

   \[
   c^2 = 1 + 1 - 2(1)(1) \cos (A+B)
   \]

   \[
   c^2 = 2 - 2\cos (A+B)
   \]

Since we have computed \( c^2 \) in two different ways, these must be equal.
\[ 2 - 2 \cos A \cos B + 2 \sin A \sin B = \cos (A + B) \]

Thus,
\[
\cos (A + B) = \cos A \cos B - \sin A \sin B
\]

And \[ \cos (A - B) = \cos (A + (-B)) \]
\[ = \cos A \cos (-B) - \sin A \sin (-B) \]
\[ = \cos A \cos B - \sin A (\sin B) \]
\[ = \cos A \cos B + \sin A \sin B. \]

Thus \[ \cos (A - B) = \cos A \cos B + \sin A \sin B \]

To derive formulas for \( \sin (A + B) \) and \( \sin (A - B) \), we can use the sum/difference formulas for cosine as well as the facts that \( \sin \theta = \cos (\frac{\pi}{2} - \theta) \) and \( \cos \theta = \sin (\frac{\pi}{2} - \theta) \).

In particular,
\[
\sin (A + B) = \sin A \cos B + \cos A \sin B
\]
and
\[
\sin (A - B) = \sin A \cos B - \cos A \sin B
\]

Next, we find an identity for \( \tan (A + B) \):

\[
\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)}
\]
\[ = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \]

Dividing all terms by \( \cos A \cos B \):
\[ = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \]
\[ = \frac{\tan A + \tan B}{1 + \tan A \tan B} \]

Thus,
\[
\tan (A + B) = \frac{\tan A + \tan B}{1 + \tan A \tan B}
\]
IV. Double Angle Formulas

Recall: \( \sin(A+B) = \sin A \cos B + \cos A \sin B \).

If \( A = B = \theta \), we get:
\[
\sin(2\theta) = 2\sin \theta \cos \theta
\]

Recall: \( \cos(A+B) = \cos A \cos B - \sin A \sin B \).

If \( A = B = \theta \), then:
\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta
\]

Then:
\[
\tan(2\theta) = \frac{\sin 2\theta}{\cos 2\theta}
= \frac{2\sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}
= \frac{2\sin \theta \cos \theta}{\cos^2 \theta}
\]

Substituting \( \cos^2 \theta = 1 - \sin^2 \theta \) gives us:
\[
\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}
\]

IV. Power Reducing Formulas

Recall: \( \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \)

and \( \cos^2 \theta + \sin^2 \theta = 1 \)

\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta
\]

Substituting \( \cos^2 \theta = 1 - \sin^2 \theta \) gives us:
\[
\cos 2\theta = 1 - 2\sin^2 \theta
\]

\[\Rightarrow 2 \sin^2 \theta = 1 - \cos 2\theta \]

\[\Rightarrow \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \]
Similarly, \( \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \)

Substituting \( \sin^2 \theta = 1 - \cos^2 \theta \), we get:

\[
\cos(2\theta) = \cos^2 \theta - [1 - \cos^2 \theta] = 2\cos^2 \theta - 1
\]

\[
2\cos^2 \theta = 1 + \cos(2\theta)
\]

\[
\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}
\]

\[\text{VII} \quad \text{Half Angle Formulas}\]

(i) From above, \( \cos^3 \theta = \frac{1 + \cos 2\theta}{2} \)

This holds for every angle \( \theta \).

Thus, if we replace \( \theta \) by \( \frac{\theta}{2} \), we get:

\[
\cos^3 \left(\frac{\theta}{2}\right) = \frac{1 + \cos \theta}{2}
\]

\[
\cos \left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos \theta}{2}}
\]

Similarly, \( \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \)

This holds for every angle \( \theta \).

Thus, if we replace \( \theta \) by \( \frac{\theta}{2} \), we get:

\[
\sin^2 \left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{2}
\]

\[
\therefore \sin \left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}
\]