Section 0.2: New facts from old

Arithmetic Operations

\[(f+g)(x) = f(x) + g(x)\]
\[(f-g)(x) = f(x) - g(x)\]
\[(fg)(x) = f(x)g(x)\]
\[\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\]

Domains

Intersection of domains of \(f\) and \(g\)

\[\text{as above, but also exclude any } x \text{ that makes } g(x) = 0.\]

\[f(x) = \frac{1}{x} \quad \text{and } g(x) = \sqrt{x + 1}\]

Define \(f+g\) and \(\frac{f}{g}\) and find their domains.

Domain \(f\): \(x \neq 0\)
Domain \(g\): \(x + 1 > 0\)
\[\Rightarrow x > -1\]

Intersection of domains:

\(x \neq 0\) and \(x > -1\)

Interval notation:

\([-1, 0) \cup (0, \infty)\]

\[f(x) = \frac{1}{x} + \sqrt{x + 1}\]

Domain \(f+g\): \([-1, 0) \cup (0, \infty)\)

\[\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{x}}{\sqrt{x + 1}} = \frac{1}{x\sqrt{x + 1}}\]

Domain: \((-1, 0) \cup (0, \infty)\) no need to exclude -1
Composition of functions

Output of one function is the input of another

\[ x \rightarrow g(x) \rightarrow f(g(x)) \rightarrow (f \circ g)(x) \]

\[ (f \circ g)(x) = f(g(x)) \]

Ex: \( f(x) = x^2 \)
    \( g(x) = x + 1 \)

Find \((f \circ g)(x)\) and \((g \circ f)(x)\):

a) \((f \circ g)(x) = f(g(x))\)
   \[ = f(x + 1) \]
   \[ = (x + 1)^2 \]

b) \((g \circ f)(x) = g(f(x))\)
   \[ = g(x^2) \]
   \[ = x^2 + 1 \]

Domain of \(f \circ g\) is:

"all \(x\) in domain of \(g\) such that \(g(x)\) is in the domain of \(f\)."

Ex: \( f(x) = x^2 + 3 \)
    \( g(x) = \sqrt{x} \)

Define \(f \circ g\) and find its domain.

\[ (f \circ g)(x) = f(g(x)) \]
\[ = f(\sqrt{x}) \]
\[ = (\sqrt{x})^2 + 3 = x + 3 \]
Be careful! Do NOT look at \((f \circ g)\) to find its domain!!

We want all \(x\) in domain of \(g\) so that \(g(x)\) is in the domain of \(f\).

i.e., we want all \(x\) in \([0, \infty)\) so that \(\sqrt{x}\) is in the \((-\infty, \infty)\).

Well, \(\sqrt{x}\) is always in \((-\infty, \infty)\) so our only restriction is that \(x \in [0, \infty)\).

NOTE: Domain is NOT all real #’s!!

\((f \circ g)(-2) = f(g(-2)) = f(\sqrt{-2}) = \text{undef!}\)

**Harder Ex:**

\[ f(x) = \sqrt{1 - x}, \quad g(x) = \sqrt{x^2 - 1} \]

**Find:**

(a) \((f \circ g)(x)\)

(b) domain of \((f \circ g)(x)\)

\[(f \circ g)(x) = f(g(x)) \]

\[= f(\sqrt{x^2 - 1}) \]

\[= \sqrt{1 - \sqrt{x^2 - 1}} \]

(b) domain \(f\):

\[1 - x \geq 0 \]

\[\Rightarrow x \leq 1 \]

\[\text{domain of } f \Rightarrow \]

\[x \leq 1 \]

\[\text{domain of } g \Rightarrow \]

\[x^2 - 1 > 0 \]

\[x^2 > 1 \]

\[x > 1 \text{ or } -1 < x \]

\[\text{domain of } g \Rightarrow \]
domain \((f \circ g)(x)\):

1. need all \(x\) in domain of \(g\) such that \(g(x)\) is in the domain of \(f\)

\[\sqrt{x^2 - 1} \leq 1\]

2. such that \(x^2 - 1 \leq 1\)

\[-\sqrt{2} \leq x \leq \sqrt{2}\]

Both requirements are satisfied when

\([-\sqrt{2}, -1] \cup [1, \sqrt{2}]\)

Domain \((f \circ g)(x)\).