Example

The accompanying figure shows the position versus time curve for an automobile over a period of time of 10 s. Use the line segments shown in the figure to estimate the instantaneous velocity of the automobile at time $t = 4$ s and again at time $t = 8$ s.

Instantaneous velocity at $t = 4$ s is mean to function at $t = 4$.

Two points on the tangent line: $(2, 0)$ and $(7, 60)$

So $V_{inst} = m_{tan} = \frac{60 - 0}{7 - 2} = 12 \frac{m}{s}$

For $t = 8$:

$V_{inst} = m_{tan} = \frac{140 - 0}{10 - 4} = \frac{70}{6} \frac{m}{s}$
Example

Consider the function \( f(x) = \frac{1}{x^2} \).

(a) Find the average rate of change, \( f_{\text{avg}} \), of \( f \) over the interval \([1,2]\).

(b) Find the instantaneous rate of change, \( f_{\text{inst}} \), of \( f \) at an arbitrary value \( x_0 \).

(c) Find the instantaneous rate of change, \( f_{\text{inst}} \), of \( f \) at \( x_0 = 1 \).
Solutions

(a) \( \text{r}_{\text{avg}} = \frac{f(2) - f(1)}{2 - 1} = \frac{\frac{1}{2} - 1}{1} = -\frac{3}{4} \)

(b) \( \text{r}_{\text{inst}} \text{ at } x_o = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_o)}{x_1 - x_o} = \lim_{x_1 \to x_0} \frac{\frac{1}{x_1^2} - \frac{1}{x_o^2}}{x_1 - x_o} = \frac{x_1^2 x_o^2}{x_1 - x_o}

\[
= \lim_{x_1 \to x_0} \frac{X_o^2 - X_1^2}{(x_1 - x_o)(x_1 x_o^2)} = \lim_{x_1 \to x_0} \frac{(x_o - x_1)(x_o + x_1)}{(x_1 - x_o)(x_1 x_o^2)}
\]

\[
= \lim_{x_1 \to x_0} \frac{-1(x_o + x_1)}{x_1^2 x_o^2} = -\frac{(x_o + x_o)}{x_o x_o^2} = -\frac{2x_o}{x_o^4} = -\frac{2}{x_o^3}
\]
OR \[ \lim_{x \to x_0} f(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \]

\[ = \lim_{h \to 0} \frac{\frac{1}{(x_0 + h)^2} - \frac{1}{x_0^2}}{h} \]

\[ = \lim_{h \to 0} \frac{x_0^2 - (x_0 + h)^2}{h(x_0 + h)^2 x_0^2} \]

\[ = \lim_{h \to 0} \frac{x_0^2 - (x_0^2 + 2x_0h + h^2)}{h(x_0 + h)^2 x_0^2} \]

\[ = \lim_{h \to 0} \frac{-2x_0h - h^2}{h(x_0 + h)^2 x_0^2} \]

\[ = \lim_{h \to 0} \frac{-2x_0 - h}{(x_0 + h)^2 x_0} \]

\[ = \frac{-2x_0}{x_0^2 x_0^2} = \frac{-2}{x_0^3} \]
(c) \( f'_{\text{inst}} \text{ at } x_0 = 1 : \frac{-2}{1^3} = -2 \)

- slope of secant line through \((1, f(1))\) and \((2, f(2))\) is \(-\frac{3}{4}\)
- slope of tangent line to \(f(x) = \frac{1}{x^2}\) at \(x=1\) is \(-2\)