Section 6.1: Area Between Curves

Recall:

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k^*) \Delta x \]

where \( f(x_k^*) \) is a sample point in the kth subinterval, and \( \Delta x = \frac{b-a}{n} \).

Suppose \( f(x) \) and \( g(x) \) are continuous and \( f(x) \geq g(x) \) on \([a, b] \).

To find the total area between \( f(x) \) and \( g(x) \), we can divide the region into \( n \) rectangles (as before).

Then, the area of the \( k \)-th rectangle (for \( k = 1, \ldots, n \)) is:

\[ \left[ f(x_k^*) - g(x_k^*) \right] \Delta x \]

where \( \Delta x = \frac{b-a}{n} \) is the width of each rectangle.

So, the total area between \( f(x) \) and \( g(x) \) on \([a, b] \) is:

\[ \lim_{n \to \infty} \sum_{k=1}^{n} \left[ f(x_k^*) - g(x_k^*) \right] \Delta x = \int_a^b \left[ f(x) - g(x) \right] dx \]
AN EQUIVALENT SETUP (USING DIFFERENTIALS):

1. Examine one of the arbitrary "really small" rectangles:

   \[ \int \frac{f(x) - g(x)}{dx} = \text{height} \]

   \[ \text{width of base} \]

2. The "really small area of this rectangle is:

\[ dA = [f(x) - g(x)] \, dx \]

3. The total area between \( f(x) \) and \( g(x) \) on \([a, b]\) is an infinite sum of these really small areas.

\[ A = \int_{a}^{b} dA \]

\[ \Rightarrow \quad A = \int_{a}^{b} [f(x) - g(x)] \, dx \]

Ex. Find the total area enclosed by \( y = x^2 \) and \( y = x + 2 \).

Points of Intersection

\[ x^2 = x + 2 \]
\[ x^2 - x - 2 = 0 \]
\[ (x - 2)(x + 1) = 0 \]
\[ x = 2 \quad x = -1 \]

Examine one arbitrary rectangle:

\[ \int (x + 2) - x^2 \]
As we slide this rectangle along the interval from $x=-1$ to $x=3$, notice that

- $y=x+2$ is always above the rectangle
- $y=x^2$ is always below the rectangle

So the area of each really small rectangle looks like: 

$$ dA = [(x+2) - x^2] \, dx $$

$$ A = \int_{-1}^{3} [(x+2) - x^2] \, dx = \cdots = \frac{9}{2} $$

Ex: Find the area enclosed by $y=x^2$ and $y=x+2$.

A) Intersection:

$y = x^2$ and $y = x+2$ 

$\Rightarrow$ $x^2 = x + 2$ 

$\Rightarrow$ $x^2 - x - 2 = 0$ 

$\Rightarrow$ $(x-2)(x+1)=0$ 

$\Rightarrow$ $x = 2, \; y = -1$ 

$\Rightarrow$ $(x=4)$ 

(b) The enclosed region extends from $x=0$ to $x=4$.

An arbitrary rectangle looks like:

As we slide this rectangle along $[0,4]$, we see that the "bottom" function is not the same for the entire interval!

$\Rightarrow$ Need $2$ definite integrals!

$A = \int_{0}^{2} (x+2) \, dx + \int_{2}^{4} x^2 \, dx$ (one for $R_1$ and one for $R_2$)
Region 1 \((R_1)\): From \(x=0\) to \(x=1\), \(x=y^2\) is both the top and bottom function.

\[
\int_{0}^{1} \sqrt{x} - (-\sqrt{x}) \, dx
\]

\[
dA = \left[ \sqrt{x} - (-\sqrt{x}) \right] dx = 2\sqrt{x} \, dx
\]

So \(A_1 = \int_{0}^{1} 2\sqrt{x} \, dx = \ldots = \frac{4}{3}\)

Region 2 \((R_2)\): From \(x=1\) to \(x=y\), the region is bounded above by \(x=y^2\) \((y=\sqrt{x})\) and bounded below by \(y=x-2\)

\[
\int_{1}^{y} \sqrt{x} - (x-2) \, dx
\]

\[
dA = \left[ \sqrt{x} - (x-2) \right] dx = \left[ \sqrt{x} - x + 2 \right] dx
\]

\(A_2 = \int_{1}^{4} (\sqrt{x} - x + 2) \, dx = \ldots = \frac{19}{6}\)

So \(A = \int_{0}^{1} 2\sqrt{x} \, dx + \int_{1}^{4} (\sqrt{x} - x + 2) \, dx = \frac{4}{3} + \frac{19}{6} = \frac{25}{6}\)
Notice:

Intersection:
\[ x = y^2 \text{ and } x = y + 2 \]
\[ \Rightarrow y = y + 2 \]
\[ \Rightarrow y^2 - y - 2 = 0 \]
\[ \Rightarrow (y - 2)(y + 1) = 0 \]
\[ \Rightarrow y = 2 \quad y = -1 \]
\[ (x = 4) \quad (x = 1) \]

If we divide the same region into horizontal rectangles

\[ \int dy \]

and slide an arbitrary rectangle up from \( y = -1 \) to \( y = 2 \), we see \( x = y^2 \) is always the left boundary and \( y = x - 2 \) (or \( x = y + 2 \)) is always the right boundary.

So:

\[ \int (y+2) - y^2 \, dy \]

\[ \Rightarrow dA = [(y+2) - y^2] \, dy = [y+2 - y^2] \, dy \]

\[ A = \int_{-1}^{2} (y+2 - y^2) \, dy \]

\[ = \cdots = \left[ \frac{9}{2} \right] \text{ Same!} \]
Ex: Find the area enclosed by \( y = \sin x \) and \( y = \cos x \) on \([0, 2\pi]\).

Intersections: \( \sin x = \cos x \) \( \Rightarrow \frac{\sin x}{\cos x} = 1 \)
\( \Rightarrow \tan x = 1 \)
\( \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \)

\[ A = \int_{0}^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x) \, dx \]

\[ = \ldots = \sqrt{2} \]

Note:

Can also be set up as \( 2 \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx \) (why?!)