defn: An ordinary differential equation (ODE) is an equation that includes the derivative of some unknown function.

defn: The order of an ODE is the highest ordered derivative in the eqn.

ex. \[ \frac{dy}{dx} = x \sqrt{x^2+4} \quad \text{1st order ODE} \]

\[ xy' = (1-x)y \quad \text{1st order ODE} \]

\[ \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 8y = 0 \quad \text{2nd order ODE} \]

defn: The solution to an ODE is a function that satisfies the eqn.

ex. Verify \( y = e^{-x} \) is a solution to \( y'' + 2y' + y = 0 \)

sln:

\[ y = e^{-x} \]

\[ y' = -e^{-x} \]

\[ y'' = e^{-x} \]

\[ y'' + 2y' + y = e^{-x} + 2(-e^{-x}) + e^{-x} = 0 \]

\[ \therefore y = e^{-x} \text{ is a soln.} \]

exercise: Verify that \( y = Ae^{2x} + Be^{-4x} \) is a solution to \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 8y = 0. \)
Solving ODEs

I. FIRST ORDER SEPERABLE EQUAS

\( F_{x,y} \) \( \frac{dy}{dx} = g(x) \)

- We “separate” the variables to the differential form:

\( h(y) \frac{dy}{dx} = g(x) dx \)

- Integrate both sides (LHS with respect to \( y \) and RHS with respect to \( x \))

\[ \int h(y) dy = \int g(x) dx \]

- This gives us

\[ H(y) = G(x) + C \]

which is an implicit solution to the ODE.

- If possible we should solve for \( y \) explicitly as a function of \( x \).

ex: solve \( \frac{dy}{dx} = 2(1+y^2)x \)

Note \( 1+y^2 \neq 0 \) \( \implies \) it is OK to divide!

Thus

\[ \frac{1}{1+y^2} \frac{dy}{dx} = 2x \]

\[ \frac{1}{1+y^2} \, dy = 2x \, dx \]

\[ \int \frac{1}{1+y^2} \, dy = \int 2x \, dx \]

\[ \tan^{-1} y = x^2 + C \]

\[ y = \tan (x^2 + C) \]

arrowed Implicit Solution

arrowed Explicit Solution
CHECK: \( y = \tan(x^2 + c) \)
\[ \Rightarrow \frac{dy}{dx} = \sec^2(x^2 + c) \cdot (2x) \]

LHS: \( \frac{dy}{dx} = 2x \cdot \sec^2(x^2 + c) \) \text{ from above} \]

RHS: \( 2(1 + y^2) \cdot x = 2(1 + \tan^2(x^2 + c)) \cdot x \]
\[ = 2(\sec^2(x^2 + c)) \cdot x \]
\[ = 2x \cdot \sec^2(x^2 + c) \]

LHS = RHS \Rightarrow y = \tan(x^2 + c) \text{ is a solution!} \]

ex solve \((4y - \cos y) \frac{dy}{dx} = 3x^2 \)
\[ \Rightarrow (4y - \cos y) \, dy = 3x^2 \, dx \]
\[ \Rightarrow \int (4y - \cos y) \, dy = \int 3x^2 \, dx \]
\[ 2y^2 - \sin y = x^3 + C \] \rightleftharpoons \text{Implicit solution}

Note: This time we can't solve for \( y \) explicitly

ex solve \( \frac{dy}{dx} = -xy \)

Case 1: if \( y \neq 0 \):
\[ \frac{1}{y} \frac{dy}{dx} = -x \]
\[ \frac{1}{y} \, dy = -x \, dx \]
\[ \int \frac{1}{y} \, dy = \int -x \, dx \]
\[ \ln |y| = - \frac{1}{2} x^2 + C \] \rightleftharpoons \text{Implicit solution}

\[ y = e^{-\frac{1}{2} x^2 + C} \]
\[ |y| = e^{-\frac{1}{2} x^2 + C} \] \text{ Positive Constant}
\[ y' = e^x e^{-\frac{1}{2}x^2} \]
\[ y = \pm e^x e^{-\frac{1}{2}x^2} \]
\[ y = Ke^{-\frac{1}{2}x^2} \quad (x \neq 0) \]

\[ y = Ke^{-\frac{1}{2}x^2} \quad \text{explicit solution.} \]

**CASE 2:** \( y = 0 \)

Then \( \frac{dy}{dx} = 0 \)

So \( \frac{dy}{dx} = -xy \) becomes \( 0 = 0 \) \( \checkmark \)

i.e., \( y = 0 \) is a solution.

Thus, the solution to the differential equation is
\[ y = Ke^{-\frac{1}{2}x^2} \quad \text{and} \quad y = 0 \quad (x \neq 0) \]

Or, if we allow \( k = 0 \) then these can be combined to
\[ y = Ke^{-\frac{1}{2}x^2} \]

ex: solve \( y' + y^2 \sin x = 0 \)

solve:
\[ \frac{dy}{dx} + y^2 \sin x = 0 \]
\[ \frac{dy}{dx} = -y^2 \sin x \]

**CASE 1:** if \( y \neq 0 \)
\[ \frac{1}{y^2} \frac{dy}{dx} = -\sin x \]
\[ \frac{1}{y^2} dy = -\sin x \, dx \]
\[ \int \frac{1}{y^2} dy = \int -\sin x \, dx \]
\[-\frac{1}{y} = \cos x + C\]
\[
y = \frac{-1}{\cos x + C} \leftarrow \text{Explicit soln}
\]

Case 2: If \( y = 0 \)
Then \( \frac{dy}{dx} = 0 \) too

So
\[
\frac{dy}{dx} + y^2 \sin x = 0 \quad \text{becomes} \quad 0 + 0 = 0 \quad \bigcirc
\]
\[
\Rightarrow \quad y = 0 \quad \text{is also a soln}
\]

Thus, the soln is
\[
y = \frac{-1}{\cos x + C} \quad \text{and} \quad y = 0
\]

Note, we can't merge the solns this time.

If Integral Curves
Suppose you're given the differential eqn
\[
\frac{dy}{dx} = 2x.
\]

When solved, the solutions are a family of functions of the form \( y = x^2 + C \)

Below are the "integral curves" with \( C = -2, -1, 0, 1, 2 \)
Notice that if you're given an initial condition such as \( y(0) = 0 \), at most one integral curve will satisfy both the differential eqn and the integral curve.

\[
\begin{align*}
\frac{dy}{dx} &= 2x \\ y(0) &= 0
\end{align*}
\]

**Initial Value Problem**

**General Solution:** \( y = x^2 + C \)

But \( y(0) = 0 \) \( \Rightarrow \) \( 0 = C + 0 \) \( \Rightarrow \) \( C = -1 \)

**Particular Soln:** \( y = x^2 - 1 \)

**Exercise:** Solve the IVP given by

\[
\begin{align*}
\frac{dy}{dx} &= x\sqrt{x^2 + 4} \\ y(-4) &= 0
\end{align*}
\]

### III. Applications and Modeling

1. Find the curve in the xy-plane which passes thru \((0, 2)\) and whose tangent line at any point \((x, y)\) has slope \( \frac{x}{y} \)

**Solu:**

- Let \( y(x) \) be the curve in the xy-plane.
- Then \( \frac{dy}{dx} = \frac{x}{y} \) because the derivative at \((x, y)\) gives the slope of the tangent line at that point.
- Also since the curve passes thru \((0, 2)\) we have an initial condition \( y(0) = 2 \)
\[ \Rightarrow \text{ IUP } \quad \begin{cases} \frac{dy}{dx} = \frac{x}{y^4} \\ y(0) = 2 \end{cases} \]

So
\[ \frac{dy}{dx} = \frac{x}{y^4} \Rightarrow y^4 \, dy = x \, dx \]

\[ \frac{y^5}{5} = \frac{x^2}{2} + C \]

General solution

\[ y = \left( \frac{5}{2} x^2 + D \right)^{\frac{1}{5}} \]

But \( y(0) = 2 \Rightarrow 2 = \left( \frac{5}{2} (0)^2 + D \right)^{\frac{1}{5}} \]

\[ \Rightarrow 2 = D^{\frac{1}{5}} \]

\[ \Rightarrow D = 32 \]

\[ \therefore y = \left( \frac{5}{2} x^2 + 32 \right)^{\frac{1}{5}} \]

Particular Solution

2. Exponential Growth Model

- models simple population growth where the rate of growth is proportional to the size of the population.
- The larger the population, the faster it grows.

Let \( y = y(t) \) be the population at time \( t \),
\[ y(0) = y_0 \] be the initial population.

Then
\[ \begin{cases} \frac{dy}{dt} = ky, \quad k > 0 \\ y(0) = y_0 \end{cases} \]

- \( k \) is the constant of proportionality called the growth constant.
- We'll solve this IUP by separation of variables
\[ \frac{dy}{dt} = ky \]
\[
\frac{dy}{dt} = ky \\
\int \frac{dy}{y} = \int k dt \\
\ln|y| = kt + C \\
\ln y = kt + C
\]

Since \( y > 0 \), don’t need abs. value.

\[
y = e^{kt+C} \quad \text{constant}
\]

\[
y = e^{kt} e^C
\]

\[
y = Ce^{kt} \quad \text{General Solution}
\]

But \( y(0) = y_0 \Rightarrow y_0 = Ce^0 \)

\[
\therefore C = y_0
\]

Thus, the particular solution to the IVP is

\[
y(t) = y_0 e^{kt}
\]

ex: An E.coli cell divides into 2 cells every 20 minutes. Let \( y = y(t) \) be the number of cells after \( t \) minutes.

a) Find a formula for \( y(t) \).

\[
y(t) = y_0 e^{kt} \quad \text{by law of exponential growth}
\]

\[
y(0) = 1 \quad \text{by we assume that we start with one cell.}
\]

Thus \( y(t) = e^{kt} \)

To solve for \( k \), we use the fact that \( y(20) = 2 \)

\[
2 = e^{20k}
\]

\[
\ln 2 = 20k
\]

\[
k = \frac{\ln 2}{20}
\]
\[ y(t) = e^{\frac{\ln 2}{20} t} = e^{\left(\frac{\ln 2}{20}\right) \frac{120}{60}} = e^{\frac{\ln 2}{2}} \]

\[ y(120) = 64 \text{ cells} \]

(b) How many cells will there be after 2 hours?

**Warning:** \( t \) is in minutes!

2 hours = 120 minutes

\[ y(120) = 2^{\frac{120}{60}} = 2^6 = 64 \text{ cells} \]

(c) How long before there are 1,000,000 cells?

\[ 1,000,000 = 2^{\frac{t}{20}} \]

\[ \ln(1,000,000) = \ln 2^{\frac{t}{20}} = \frac{t}{20} \ln 2 \]

\[ t = 20 \ln \left(\frac{1,000,000}{2}\right) \approx 399 \text{ minutes} \]

(3) Logistic Model

- A more realistic population model where the population eventually levels off to a carrying capacity \( L \) of the system.

\[
\begin{align*}
\frac{dy}{dt} &= k \left(1 - \frac{y}{L}\right)y, \quad k > 0 \\
y(0) &= y_0
\end{align*}
\]

Note: If \( \frac{y}{L} \) is small, then \( \frac{dy}{dt} \approx ky \)
and the population grows like the exponential growth model.
If \( y = L \), then \( \frac{dy}{dt} = 0 \)

If \( y > L \), then \( \frac{dy}{dt} < 0 \)
(The population has grown too large.
(And thus decreases)

**Exercise:** Solve the IVP

**Hint:** use partial fractions

**Solution:**

\[
Y(t) = \frac{y_0 L}{y_0 + (L - y_0)e^{-kt}}
\]

**Note:** as \( t \to \infty \), \( y(t) \to L \)

4. **Exponential Decay Model**

- Rate of decay is proportional to the amount of substance present.
- \( y = y(t) = \text{amount of substance remaining at time } t \)
- \( y(0) = y_0 = \text{initial amount of substance} \)

\[
\begin{align*}
\frac{dy}{dt} &= -Ky, \quad K > 0 \\
y(0) &= y_0
\end{align*}
\]

**Exercise:** Show that the solution to this model is:

\[
Y(t) = y_0 e^{-kt}
\]

Ex: How long before half of the original substance remains?

If half remains, then \( y = \frac{1}{2} y_0 \)

Thus \( y = y_0 e^{-kt} \) becomes:
\[ \frac{1}{2}y_0 = y_0 e^{-kt} \]
\[ \frac{1}{2} = e^{-kt} \]
\[ \ln \left( \frac{1}{2} \right) = \ln (e^{-kt}) \]
\[ \ln \left( \frac{1}{2} \right) = -kt \]
\[ t = -\frac{1}{k} \ln \left( \frac{1}{2} \right) = \frac{\ln \left( \frac{1}{2} \right)}{k} = \frac{\ln 2}{k} \]

Thus, the \( \frac{1}{2} \) life is given by

\[ t = \frac{\ln 2}{k} \]

**Note:** this does not depend on \( y_0 \)!

**Exercise:** Prove that in the exponential growth model, \( t = \frac{\ln 2}{k} \) is the amount of time it takes for the population to double. This is called the doubling time of the population.

---

**Exercise:** Suppose 30% of a radioactive substance decays in 5 years. Find the half-life of the substance.

**Note:**

Given 30% decays, \( 70\% \) remains

\[ y = y_0 e^{-kt} \]
where \( y(5) = 0.7y_0 \).

So, we can solve for the decay constant \( k \).

\[ 0.7y_0 = y_0 e^{-5k} \]
\[ 0.7 = e^{-5k} \]
\[ \ln(0.7) = \ln(e^{-5k}) \]
\[ \ln(0.7) = -5k \]
\[ k = \frac{\ln(0.7)}{-5} \]

Thus, the half-life is \( t = \frac{\ln 2}{k} = \frac{\ln 2}{\ln(0.7)/-5} \approx 9.7 \text{ yrs} \)
Given the half life of carbon-14 is 5730 years, find the decay constant.

\[ t_{\text{half}} = \frac{\ln 2}{k} \implies k = \frac{\ln 2}{t_{\text{half}}} \]

\[ \therefore k = \frac{\ln 2}{5730} \approx 0.000121 \]

Ex (Shroud of Turin)

When tested in 1988, fibers in the cloth contained 93% of the original carbon 14. Determine the year of origin of the cloth.

\[ y = y_0 e^{-kt} \implies y = y_0 e^{-0.000121 t} \]

To find the time corresponding to 1988 when 93% remains, we set \( \frac{y}{y_0} = 0.93 \)

Thus \( \frac{y}{y_0} = e^{-0.000121 t} \)

\[ 0.93 = e^{-0.000121 t} \]

\[ \therefore t = -\frac{\ln(0.93)}{0.000121} \approx 600 \text{ years} \]

So, the year of origin is approximately 1988 - 600 = 1388

6 Newton’s Law of Cooling

Fact: The rate at which the temp of a cooling object decreases is proportional to the difference between the temp of the object and the temp of the surrounding environment.
The same is true for the rate at which a warming object increases.

Let $T(t) = \text{temp of obj at time } t$.
$T(0) = T_0 = \text{initial temp of object}$
$T_e = \text{constant temp of surrounding environment}$

Then
\[ \begin{cases} \frac{dT}{dt} = k(T - T_e), & k < 0 \\ T(0) = T_0 \end{cases} \]

Note: when an obj cools off $T > T_e \iff k < 0$

else $\frac{dT}{dt} > 0$

when an obj warms up $T_e > T \iff k < 0$

else $\frac{dT}{dt} < 0$

Alternatively
\[ \begin{cases} \frac{dT}{dt} = k(T_e - T), & k > 0 \\ T(0) = T_0 \end{cases} \]

Solve: $T(t) = T_e + (T_0 - T_e)e^{kt}$ $(k < 0)$

or

$T(t) = T_e + (T_0 - T_e)e^{-kt}$ $(k > 0)$