MATH 200 - SAMPLE Exam 2 - 5/25/2012

NAME: Key

SECTION: __________________________

Directions:

- For the free response questions, you must show all work to receive full credit. Cross off any work that you do not want graded.

- You have fifty (50) minutes to complete this exam. When time is called, STOP WRITING IMMEDIATELY.

- You may not use any electronic devices including (but not limited to) calculators, cell phones, or iPods.

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Multiple Choice

**Directions:** Circle the letter of the best answer. Make sure your circles include just one letter. Any answer that looks like two letters will be marked incorrect. Each multiple choice problem is worth 5 points.

1. For the function \( f(x, y, z) = xyz + yx^2 + xy^2 \) which of the following is a unit vector \( \mathbf{u} \) in which the function is decreasing most rapidly at the point \( Q(-1, 3, 0) \).
   - (a) \( \left\langle \frac{3}{\sqrt{42}}, -\frac{5}{\sqrt{42}}, -\frac{3}{\sqrt{42}} \right\rangle \)
   - (b) \( \left\langle -\frac{3}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{3}{\sqrt{42}} \right\rangle \)
   - (c) \( \left\langle -\frac{3}{\sqrt{42}}, -\frac{5}{\sqrt{42}}, \frac{3}{\sqrt{42}} \right\rangle \)
   - (d) \( (-3, 5, 3) \)
   - (e) \( (3, -5, -3) \)

   **At \( Q, f(x, y, z) \) decreases most rapidly in the direction of \(-\nabla f(Q)\).**

   \[ \mathbf{u} = -\frac{\nabla f(Q)}{\| \nabla f(Q) \|} \]

2. Which of the following is the equation for the tangent plane to the surface at point \( P \).
   \[ 2x^2y^2 + 3xz^2 = -4; \quad P(-1, -2, 2) \]
   - (a) \( -x - 2y + 2z = 9 \)
   - (b) \( x + 2y + 3z = 1 \)
   - (c) \( x + 2y + 3z = 11 \)
   - (d) \( 2x + 4y - 3z = 16 \)
   - (e) \( -4x - 8y - 12z = 44 \)

3. The function \( f(x, y) = 3x^2 + 2y^3 + 12x - 6y + 5 \) has two critical points: \((-2, -1)\) and \((-2, 1)\). Which of the following is true?
   - (a) There is a relative (local) maximum at \((-2, 1)\)
   - (b) There is a relative (local) maximum at \((-2, 1)\)
   - (c) There is a relative (local) minimum at \((-2, -1)\)
   - (d) There is a relative (local) minimum at \((-2, 1)\)
   - (e) There is a saddle point at \((-2, 1)\)

   Let \((x_0, y_0)\) be a crit. pt.
   \( D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \)
   - If \( D(x_0, y_0) < 0 \), then \( f \) has a saddle pt.
   - If \( D(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) > 0 \) then local min.
   - If \( D(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) < 0 \) then local max.
4. An equation of the tangent plane to the surface \( f(x, y, z) = k \) at \((1, -4, 1)\) is
\[
3(x - 1) + 5(y + 4) - 2(z - 1) = 0
\]
Which of the following vectors must be tangent to the surface \( f(x, y, z) = k \) at \((1, -4, 1)\)?
(a) \((-1, 4, -1)\)
(b) \((6, 10, -4)\)
(c) \((-3, -5, 2)\)
(d) \((-1, 1, -1)\)
(e) \((1, -1, -1)\) \[\langle 3, 5, -2 \rangle \text{ is } \perp \text{ to } f.\]
\[\text{A tangent vector must then be } \perp \text{ to } \langle 3, 5, -2 \rangle.\]
\[\text{Choice } \hat{e} \cdot \langle 3, 5, -2 \rangle = 0.\]

5. A particle moves along the intersection of the elliptic paraboloid \( z = x^2 + 2y^2 \) and the plane \( x = 2 \). At the moment when the particle is at \((2, 1, 6)\), which of the following is the rate that \( z \) is changing with respect to \( y \)?
(a) 4
(b) 6
(c) 8
(d) 12
(e) 16
\[
\frac{\partial z}{\partial y} \bigg|_{(2, 1, 6)} = 211, 6).
\]

6. Let \( f(x, y) = x^2y - y^2 - z^3 \). Which of the following is the directional derivative of \( f(x, y, z) \) at \( P(-1, 1, 1) \) in the direction of \( \hat{d} = \frac{1}{\sqrt{3}} \hat{i} - \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k} \)?
(a) \(-2 \hat{i} - \hat{j} - 3 \hat{k}\)
(b) \(-\frac{2}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} - \frac{3}{\sqrt{3}} \hat{k}\)
(c) \(-4\)
(d) \(-\frac{2}{\sqrt{3}}\)
(e) \(-\frac{4}{\sqrt{3}}\)
\[\text{D}_\hat{d} f(-1, 1, 1) = \nabla f(-1, 1, 1) \cdot \hat{u}\]
\[\text{where } \hat{u} = \frac{\hat{d}}{||\hat{d}||}.\]
7. Which of the following is a sketch of the domain of \( f(x, y) = \ln(xy - 1) + e^{x^2y} - y^3? \)

(a) 

(b) 

(c) 

(d) 

(e) 

all points in \( \mathbb{R}^2 \) such that:

\[
\begin{align*}
xy - 1 &> 0 \\
xy &> 1 \\
y &> \frac{1}{x}
\end{align*}
\]

8. Let \( S \) be the surface \( z = x^2 + y^2 \) and let \( P(1, 1, 2) \) be a point on the graph of \( S \). What is the angle between the normal line to the surface \( S \) at the point \( P \) and the tangent line to the surface \( S \) at the point \( P \).

(a) \( 0 \)

(b) \( \frac{\pi}{4} \)

(c) \( \frac{\pi}{3} \)

(d) \( \frac{\pi}{2} \)

(e) \( \frac{5\pi}{6} \)
9. Which of the following is NOT true?

(a) Let \( k \) be a constant. Then \( \nabla f(x_0, y_0) \) is normal to \( f(x_0, y_0) = k \) at the point \( P(x_0, y_0) \).

(b) The directional derivative of \( f(x, y) \) at the point \( P(x_0, y_0) \) is largest in the direction of \( \nabla f(x_0, y_0) \).

(c) If \( f(x, y) \) is differentiable at \( P(x_0, y_0) \) and \( \nabla f(x_0, y_0) = 0 \), then \( f(x, y) \) has a critical point at \( P \).

(d) If \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \) and \( f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0 \), then \( f \) has a relative (local) maximum at \( (x_0, y_0) \)

(e) Let \( f \) be a function of two variables. If \( f_{xy} \) and \( f_{yx} \) are continuous on some open disk, then \( f_{xy} = f_{yx} \) on that disk.

10. Using a formula from geometry, evaluate \( \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2} \, dy \, dx \). (HINT: volume)

(a) 0

(b) \( \frac{9\pi}{2} \)

(c) \( 9\pi \)

(d) 18\pi

(e) 36\pi

This represents the volume under \( z = \sqrt{9-x^2-y^2} \) over \( \text{over } x \)

11. Let \( z = z(x, y) \), where \( x = x(s, t) \) and \( y = y(s, t) \). Which of the following is \( \frac{\partial z}{\partial s} \)?

(a) \( \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} \)

(b) \( \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \)

(c) \( \frac{\partial z}{\partial x} \frac{\partial y}{\partial s} + \frac{\partial z}{\partial x} \frac{\partial y}{\partial s} \)

(d) \( \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} \frac{\partial y}{\partial s} \)

(e) \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \)

Chain Rule.
Free-Response

**Directions** For each of the following problems you must show all of your work to get full credit. Simplify when appropriate. Each problem is worth 15 points.

12. Use implicit differentiation to find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

\[
yx^2 + z \cos \left( \frac{xy}{z} \right) = 0
\]

\[
\frac{\partial}{\partial x} \left[ yx^2 + z \cos \left( \frac{xy}{z} \right) \right] = \frac{\partial}{\partial x} (0)
\]

\[
2xy + z \left[ -\sin (xyz^{-1}) \left( x - yz^{-2} \frac{\partial z}{\partial x} \right) + yz^{-1} \right] + \cos (xyz^{-1}) \frac{\partial z}{\partial x} = 0
\]

\[
2xy + z \sin (xyz^{-1}) xyz^{-2} \frac{\partial z}{\partial x} + y \sin (xyz^{-1}) + \cos (xyz^{-1}) \frac{\partial z}{\partial x} = 0
\]

\[
\frac{\partial z}{\partial x} = \frac{-2xy + y \sin (xyz^{-1})}{z \sin (xyz^{-1}) xyz^{-2} + \cos (xyz^{-1})}
\]

\[
\frac{\partial z}{\partial x} = \frac{-2xy + y \sin (xyz^{-1})}{xyz^{-1} \sin (xyz^{-1}) + \cos (xyz^{-1})}
\]

\[
\frac{\partial z}{\partial y} \text{ is similar!}
\]
13. Compute parametric equations of the line tangent to the curve of intersection of the paraboloid \( z = \frac{1}{2} (x^2 + y^2) \) and the ellipsoid \( x^2 + 2y^2 + z^2 = 4 \) at \((-1, 1, 1)\).

\[
\frac{1}{2}x^2 + \frac{1}{2}y^2 = z
\]

\( f(x, y, z) \)

\( \nabla f(x, y, z) \perp \) to our surface at \((x, y, z)\) is our surface is a level surface of \( f \).

\[
\nabla f(x, y, z) = \langle x, y, -1 \rangle
\]

\( \nabla f(-1, 1, 1) = \langle -1, 1, -1 \rangle \).

Similarly, let \( g(x, y, z) = x^2 + 2y^2 + z^2 \)

Then \( \nabla g(x, y, z) = \langle 2x, 4y, 2z \rangle \)

\( \nabla g(-1, 1, 1) = \langle -2, 4, 2 \rangle \).

Since \( \nabla f(-1, 1, 1) \) and \( \nabla g(-1, 1, 1) \) are \( \perp \) to the curve of intersection, a vector tangent to the curve of intersection is

\[
\vec{v} = \nabla f(-1, 1, 1) \times \nabla g(-1, 1, 1) = \begin{vmatrix} i & j & k \\ -1 & 1 & -1 \\ -2 & 4 & 2 \end{vmatrix} = \langle 6, 4, -2 \rangle
\]

\[
\begin{align*}
\begin{cases}
x &= -1 + 6t \\
y &= 1 + 4t \\
z &= 1 - 2t
\end{cases}
\end{align*}
\]
14. Change the order of integration to $dydx$. Then evaluate the integral.

\[ \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^2 + 1} \, dx \, dy \]

$x = \sqrt{y} \implies y = x^2$

\[ K = \int_0^1 \sqrt{x^2 + 1} \, y \bigg|_{y=0}^{y=x^2} \, dx \]

\[ = \int_0^1 \sqrt{x^2 + 1} \, x^2 \, dx \]

Let $u = x^2 + 1$

$du = 2x \, dx \implies \frac{1}{2} \, du = x \, dx$

$x = 0 \implies u = 1$

$x = 1 \implies u = 2$

\[ = \frac{1}{3} \int_1^2 u^{3/2} \, du \]

\[ = \left. \frac{1}{3} \frac{2}{3} u^{3/2} \right|_{u=1}^{u=2} \]

\[ = \frac{2}{9} \left( 2^{3/2} - 1 \right) \]