Final Exam Outcomes List – Even Numbered Problems – Solutions

Chapter 11.8

24. This is the plane \( x = 2 \)
30. This is the plane \( z = 2 \).

Chapter 14.4

26. \( x = 3 \cos \theta, y = 3 \sin \theta, z = \cot \phi \)
30. \( z = x^2 + y^2, 0 \leq z \leq 4 \), part of a circular paraboloid

Chapter 14.5

16. \( V = \int_0^1 \int_0^{1-x} \int_0^{\sqrt{y}} 1 \, dz \, dy \, dx = \frac{4}{15} \)

Chapter 14.6

30. (a) The sphere and cone intersect in a circle of radius \( \rho_0 \sin \phi_0 \),

\[
V = \int_{\theta_1}^{\theta_2} \int_0^{\rho_0 \sin \phi_0} \int_0^{\sqrt{\rho_0^2 - r^2}} r \, dz \, dr \, d\theta = \int_{\theta_1}^{\theta_2} \int_0^{\rho_0 \sin \phi_0} \left( r \sqrt{\rho_0^2 - r^2} - r^2 \cot \phi_0 \right) dr \, d\theta
\]

\[
= \int_{\theta_1}^{\theta_2} \frac{1}{3} \rho_0^3 (1 - \cos^3 \phi_0 - \sin^3 \phi_0 \cot \phi_0) \, d\theta = \frac{1}{3} \rho_0^3 (1 - \cos^3 \phi_0 - \sin^3 \phi_0 \cos \phi_0) (\theta_2 - \theta_1)
\]

\[
= \frac{1}{3} \rho_0^3 (1 - \cos \phi_0)(\theta_2 - \theta_1).
\]

(b) From part (a), the volume of the solid bounded by \( \theta = \theta_1, \theta = \theta_2, \phi = \phi_1, \phi = \phi_2 \), and \( \rho = \rho_0 \) is

\[
\rho = \rho_0 \text{ is } \frac{1}{3} \rho_0^3 (1 - \cos \phi_2)(\theta_2 - \theta_1) - \frac{1}{3} \rho_0^3 (1 - \cos \phi_1)(\theta_2 - \theta_1) = \frac{1}{3} \rho_0^3 (\cos \phi_1 - \cos \phi_2)(\theta_2 - \theta_1)
\]

so the volume of the spherical wedge between \( \rho = \rho_1 \) and \( \rho = \rho_2 \) is

\[
\Delta V = \frac{1}{3} \rho_2^3 (\cos \phi_1 - \cos \phi_2)(\theta_2 - \theta_1) - \frac{1}{3} \rho_1^3 (\cos \phi_1 - \cos \phi_2)(\theta_2 - \theta_1)
\]

\[
= \frac{1}{3} (\rho_2^3 - \rho_1^3) (\cos \phi_1 - \cos \phi_2)(\theta_2 - \theta_1)
\]

(c) \( \frac{d}{d\phi} \cos \phi = -\sin \phi \) so from the Mean-Value Theorem \( \cos \phi_2 - \cos \phi_1 = -(\phi_2 - \phi_1) \sin \phi^* \) where \( \phi^* \) is between \( \phi_1 \) and \( \phi_2 \). Similarly \( \frac{d}{d\rho} \rho^3 = 3 \rho^2 \) so \( \rho_2^3 - \rho_1^3 = 3 \rho^2 (\rho_2 - \rho_1) \) where \( \rho^* \) is between \( \rho_1 \) and \( \rho_2 \). Thus \( \cos \phi_1 - \cos \phi_2 = \sin \phi^* \Delta \phi \) and \( \rho_2^3 - \rho_1^3 = 3 \rho^2 \Delta \rho \) so \( \Delta V = \rho^2 \sin \phi^* \Delta \rho \Delta \phi \Delta \theta \).

Chapter 14.7

22. \( \frac{1}{2} \left( e^4 - e - 3 \right) \)