Chapter 13.4: Local Linear Approximations: Differentials

Review:

\[ y = f(x) \]

Tangent line to \( y = f(x) \) at \( x_0 \)

Equation:

\[ y - y_0 = m(x - x_0) \]

\[ y - f(x_0) = f'(x_0)(x - x_0) \]

\[ L(x) = f(x_0) + f'(x_0)(x - x_0) \]

\( L(x) \) is the local linear approximation to \( f \) at \( x_0 \)

So when \( x \) is near \( x_0 \), \( f(x) \approx L(x) \)

E.g. \( \sqrt[3]{x} \approx \frac{1}{3} x + \frac{2}{3} \) near \( x_0 = 1 \)

(See Calc I notes if you need more details)

So \( \sqrt[3]{1.1} \approx \frac{1}{3} (1.1) + \frac{2}{3} = 1.033333 \)  
Calculator: \( \sqrt[3]{1.1} \approx 1.03228 \)
New stuff:

\[ L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \]

is the **local linear approximation** to \( f(x, y) \) at \( (x_0, y_0) \)

[Note: We will see that \( z = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \)

is the equation of the tangent plane to \( f(x, y) \) at \( (x_0, y_0) \)]

So when \( (x, y) \) is near \( (x_0, y_0) \), \( f(x, y) \approx L(x, y) \)
Example: Approximate $\sqrt{(3.04)^2 + (3.98)^2}$

So $f(x, y) = \sqrt{x^2 + y^2}$, $(x_0, y_0) = (3, 4)$

$L(x_1, y_1) = f(3, 4) + f_x(3, 4)(x-3) + f_y(3, 4)(y-4)$

Now $f(3, 4) = \sqrt{3^2 + 4^2} = 5$

$f_x = \frac{1}{2\sqrt{x^2 + y^2}} (2x) = \frac{x}{\sqrt{x^2 + y^2}} \Rightarrow f_x(3, 4) = \frac{3}{5}$

$f_y = \frac{1}{2\sqrt{x^2 + y^2}} (2y) = \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow f_y(3, 4) = \frac{4}{5}$

So $L(x_1, y_1) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$

$L(3.04, 3.98) = 5 + \frac{3}{5}(0.04) + \frac{4}{5}(-0.02) = 5.008$

Calculator: $\sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$
Note: All of the above can be applied to functions of three variables, e.g., \( W = f(x, y, z) \)

\[ L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \]

is the local linear approximation to \( f(x, y, z) \) at \( (x_0, y_0, z_0) \)

[So if \( (x, y, z) \) stays close to \( (x_0, y_0, z_0) \), then \( f(x, y, z) \approx L(x, y, z) \).]

\[ dw = f_x(x, y, z) \, dx + f_y(x, y, z) \, dy + f_z(x, y, z) \, dz \]

is the total differential.