Direction: Show all work.

1. You are to pay $1 to play a game that consists of drawing one ticket at random from a box of numbered tickets as shown below. You win the amount (in dollars) of the number on the ticket that you draw. Calculate the expected value and variance of your net gain.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>P(X) = P(Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>1/3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1/3</td>
</tr>
</tbody>
</table>

$E(Y) = (-1) \left( \frac{1}{3} \right) + (0) \left( \frac{1}{3} \right) + (1) \left( \frac{1}{3} \right) = 0$

$E(Y^2) = (-1)^2 \left( \frac{1}{3} \right) + (0^2) \left( \frac{1}{3} \right) + (1^2) \left( \frac{1}{3} \right) = \frac{2}{3}$

$V(Y) = E(Y^2) - (E(Y))^2 = \frac{2}{3} - 0^2 = \frac{2}{3}$

2. Let $X$ be a random variable with $E(X) = 1$ and $V(X) = 5$. Compute:
   a) $E[(3 + x)^2]$
   b) $V(3x + 4)$

   a) $E[(3 + x)^2] = E(9 + 6x + x^2) = 9 + 6E(x) + E(x^2)$

   b) $V(3x + 4) = 3^2 V(X) = 9(5) = 45$
3. The suicide rate in a certain state is 1 suicide per 100,000 inhabitants (per month). A particular city in the state has 400,000 inhabitants.
   a) Find the probability that there will be at least 3 suicides in the city in a given month.
   b) What is the probability that there will be at least two months during the year in which the city will have at least 3 suicides?

   a) 1 suicide per 100,000 $\rightarrow$ 4 suicides per 400,000

   Let $X$ = # Suicides in the city in a given month

   Then $X \sim$ Poisson ($\lambda=4$)

   $f(x) = \begin{cases} \frac{4^x e^{-4}}{x!} & x=0,1,2,... \\ 0 & \text{otherwise} \end{cases}$

   $P(X \geq 3) = 1 - P(X < 3) = 1 - P(X=0) - P(X=1) - P(X=2)$

   $= 1 - e^{-4} - 4e^{-4} - \frac{4^2 e^{-4}}{2!}$

   $= 0.762$ (boxed)

   b) Let $Y$ = # of months out of 12 where the city has $\geq 3$ suicides.

   $Y \sim$ Binomial ($n=12, \ P=0.762$)

   $f(y) = \binom{12}{y} (0.762)^y (0.238)^{12-y} \quad y=0,1,...,12$

   $= 0, -5.$

   $P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$

   $= 0.99999$ (boxed)
4. A firm sells four items randomly from a large lot of which it is known that 12% is defective. Let $X$ denote the number of defectives among the four sold. The purchaser will return any defective items for repair and the cost associated with repair is $C = 2X^2 + X + 3$. Find the expected cost.

Let $X = \#$ defective items from the 4

$$X \sim \text{Binomial}(n = 4, p = 0.12)$$

$$f(x) = \begin{cases} \binom{4}{x} (0.12)^x (0.88)^{4-x} & x = 0, 1, \ldots, 4 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = np = (4)(0.12) = 0.48$$

$$V(X) = npq = (4)(0.12)(0.88) = 0.4224$$

$$V(X) = E(X^2) - (E(X))^2$$

$$0.4224 = E(X^2) - (0.48)^2 \quad \Rightarrow \quad E(X^2) = 0.6528$$

Then

$$E(C) = E(2X^2 + X + 3)$$

$$= 2E(X^2) + E(X) + 3$$

$$= 2(0.6528) + 0.48 + 3 \approx 4.79$$
5. People with $O^-$ blood are called universal donors because they may give blood to anyone without risking incompatibility due to blood type factors (A or B). Of the persons donating blood at a clinic, 9% have $O^-$ blood.

a) Find the probability that the first $O^-$ donor is found after blood typing five people who are not $O^-$. 

b) What is the expected number of blood tests needed to obtain the first $O^-$ donor?

\[ a) \text{ Let } X = \# \text{ blood tests not resulting in } O^- \text{ before } 1^\text{st} \text{ one } \rightarrow O^- \]

Then \( X \sim \text{Geometric} \ (p = 0.09) \)

\[ f(x) = \begin{cases} (0.91)^x (0.09) & X = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \]

\[ P(X = 5) = (0.91)^5 (0.09) = 0.556 \]

\[ b) \text{ Expected } \# \text{ of non- } O^- \text{ is } \frac{0.91}{0.09} = 10.11 \]

\[ \therefore \text{ Expected } \# \text{ of blood tests is } 11.11 \]
6. Compute the moment generating function for a continuous random variable \( X \) which is uniformly distributed on \((a, b)\).

\[
M_X(t) = E(e^{tx}) = \int_a^b e^{tx} \frac{1}{b-a} \, dx
\]

\[
= \frac{1}{b-a} \left[ e^{tx} \right]_a^b
\]

\[
= \frac{e^{tb} - e^{ta}}{b-a}
\]

7. You arrive at a bus stop at 10 o’clock, knowing that the bus will arrive some time uniformly distributed between 10:00 and 10:30. What is the probability that you will have to wait longer than 10 minutes?

Let \( X \) = minutes waited

\( X \) uniformly distributed on \([0, 30]\)

\[
f(x) = \begin{cases} \frac{1}{30} & x \in [0, 30] \\ 0 & \text{otherwise} \end{cases}
\]

\[
P(X > 10) = \int_{10}^{30} \frac{1}{30} \, dx = \frac{20}{30} = \frac{2}{3}
\]
8. Extensive experience with fans of a certain type used in diesel engines has suggested that the exponential distribution provides a good model for the time until failure. Suppose the mean time until failure is 25,000 hours. Calculate the probability that a randomly selected fan will last at between 20,000 and 30,000 hours.

SEE HW.

9. Let \( X \) be a normal random variable with a mean of 12 and a variance of 4. Find the value of \( c \) such that \( P(X > c) = 0.1 \).

\[
P(X > c) = 0.1
\]
\[
1 - P(X \leq c) = 0.1
\]
\[
P(X \leq c) = 0.9
\]
\[
P\left(\frac{X - 12}{2} \leq \frac{c - 12}{2}\right) = 0.9
\]
\[
P\left(Z \leq \frac{c - 12}{2}\right) = 0.9
\]
\[
\frac{c - 12}{2} = 1.285
\]
\[
c = 14.57
\]
10. An examination is often regarded as good (i.e., has a valid grade spread) if the test scores of those taking it can be approximated by a normal distribution. The instructor uses the test scores to estimate parameters $\mu$ and $\sigma^2$. Then she assigns grades according to the following chart:

<table>
<thead>
<tr>
<th>Grade</th>
<th>Score Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Students who score greater than $\mu + \sigma$</td>
</tr>
<tr>
<td>B</td>
<td>Students who score between $\mu$ and $\mu + \sigma$</td>
</tr>
<tr>
<td>C</td>
<td>Students who score between $\mu - \sigma$ and $\mu$</td>
</tr>
<tr>
<td>D</td>
<td>Students who score between $\mu - 2\sigma$ and $\mu - \sigma$</td>
</tr>
<tr>
<td>F</td>
<td>Students who score below $\mu - 2\sigma$</td>
</tr>
</tbody>
</table>

This is sometimes called “grading on the curve.”

a) What is the probability that a given student will earn an A?

b) What is the probability that a given student will earn a B?

c) What is the probability that a given student will earn a D?

\[
P(X > \mu + \sigma) = P\left(\frac{X-\mu}{\sigma} > \frac{\mu+\sigma-\mu}{\sigma}\right)
\]

\[
= P(Z > 1)
\]

\[
= 1 - P(Z \leq 1)
\]

\[
= 1 - 0.8413 = 0.1587
\]

\[
P(\mu < X < \mu + \sigma) = P\left(\frac{\mu - \mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\mu + \sigma - \mu}{\sigma}\right)
\]

\[
= P(0 < Z \leq 1)
\]

\[
= P(Z \leq 1) - P(Z = 0)
\]

\[
= 0.8413 - 0.5 = 0.3413
\]

\[
P(\mu - 2\sigma \leq X \leq \mu - \sigma)
\]

\[
= P\left(\frac{\mu - 2\sigma - \mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{\mu - \sigma - \mu}{\sigma}\right)
\]

\[
= P(-2 \leq Z \leq -1)
\]

\[
= P(Z \leq -1) - P(Z \leq -2)
\]

\[
= P(Z \geq 1) - P(Z > 2)
\]

\[
= (1 - P(Z \leq 1)) - (1 - P(Z \leq 2))
\]

\[
P(Z \leq 2) - P(Z \leq 1)
\]

\[
= 0.7772 - 0.5 = 0.2772
\]

\[
P(Z \leq 2) - P(Z \leq 1)
\]

\[
= 0.7772 - 0.5 = 0.2772
\]

\[
P(Z \leq 2) - P(Z \leq 1)
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P(Z \leq 2) - P(Z \leq 1)
\]

\[
= 0.7772 - 0.5 = 0.2772
\]
Extra Credit

The Rayleigh Distribution is a continuous distribution with the following pdf:

\[
f(x) = \begin{cases} 
\frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & x \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

where \( \theta > 0 \).

a) Show that this is a valid pdf.

b) Compute \( E(X) \) and \( V(X) \).
   (Your answers will involve \( \theta \))