### 4.3 Exercises

For Exercises 1 through 8, let $R$ be the relation whose digraph is given in Figure 4.16.

**Figure 4.16**

1. List all paths of length 1.
2. (a) List all paths of length 2 starting from vertex 2.
   (b) List all paths of length 2.
3. (a) List all paths of length 3 starting from vertex 3.
   (b) List all paths of length 3.
4. Find a cycle starting at vertex 2.
5. Find a cycle starting at vertex 6.
6. Draw the digraph of $R^2$.
7. Find $M_{R^1}$.
8. (a) Find $R^w$.
   (b) Find $M_{R^w}$.

For Exercises 9 through 16, let $R$ be the relation whose digraph is given in Figure 4.17.

**Figure 4.17**

9. List all paths of length 1.
10. (a) List all paths of length 2 starting from vertex c.
    (b) Find all paths of length 2.
11. (a) List all paths of length 3 starting from vertex a.
    (b) Find all paths of length 3.
12. Find a cycle starting at vertex c.
13. Find a cycle starting at vertex d.
14. Find a cycle starting at vertex a.
15. Draw the digraph of $R^2$.
16. Find $M_{R^2}$. Is this result consistent with the result of Exercise 15?
17. (a) Find $M_{R^w}$.
    (b) Find $R^w$.
18. Let $R$ and $S$ be relations on a set $A$. Show that
    
    $M_{R\cap S} = M_R \cap M_S$.

19. Let $R$ be a relation on a set $A$ that has $n$ elements. Show that $M_{R^n} = M_{R^w} \cap I_n$, where $I_n$ is the $n \times n$ identity matrix.

In Exercises 20 through 25, let $R$ be the relation whose digraph is given in Figure 4.18.

**Figure 4.18**

20. If $\pi_1: 1, 2, 4, 3$ and $\pi_2: 3, 5, 6, 4$, find the composition $\pi_1 \circ \pi_2$.
21. If $\pi_1: 1, 7, 5$ and $\pi_2: 5, 6, 7, 4, 3$, find the composition $\pi_1 \circ \pi_2$.
22. If $\pi_1: 3, 4, 5, 6, 7$ and $\pi_2: 6, 7, 4, 3, 5$, find the composition $\pi_1 \circ \pi_2$.
23. If $\pi_1: 2, 3, 5, 6, 7$ and $\pi_2: 7, 5, 6, 4$, find the composition $\pi_1 \circ \pi_2$.
24. Find two cycles of length at least 3 in the relation $R$.
25. Find a cycle with maximum length in the relation $R$.
26. Let $A = \{1, 2, 3, 4, 5\}$ and $R$ be the relation defined by $a R b$ if and only if $a < b$.
   (a) Compute $R^2$ and $R^3$.
   (b) Complete the following statement: $a R^3 b$ if and only if __________
   (c) Complete the following statement: $a R^3 b$ if and only if __________
27. By Theorem 1, \( M_R \otimes M_R = M_{2R} \) so that \( M_R \otimes M_R \) shows where there are paths of length 2 in the digraph of \( R \). Let

\[
M_R = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}.
\]

What does \( M_R \cdot M_R \) show? Justify your conclusion.

28. Is it possible to generalize the results of Exercise 27? For example, does \((M_R)^2\) tell us anything useful about \( R \)?

29. Complete the following. The proof of Theorem 1 is based on proof of matrices.

30. (a) What about the statement of Theorem 2 indicates that an induction proof is appropriate?

(b) What is the central idea of the induction step in the proof of Theorem 2?

31. Let \( D \) be the digraph of a finite relation \( R \). Show that if there are no cycles in \( D \), then at least one vertex has out-degree 0.

32. Draw a digraph with six vertices that has exactly one path of length 6 and exactly six paths of length 1.

33. Juan and Nils have each drawn a digraph to represent the relation \( R \). The digraphs do not "look" alike. How would you determine if the digraphs both represent \( R \) correctly?

4.4 Properties of Relations

In many applications to computer science and applied mathematics, we deal with relations on a set \( A \) rather than relations from \( A \) to \( B \). Moreover, these relations often satisfy certain properties that will be discussed in this section.

- Reflexive and Irreflexive Relations

A relation \( R \) on a set \( A \) is reflexive if \((a, a) \in R\) for all \( a \in A \), that is, if \( a R a \) for all \( a \in A \). A relation \( R \) on a set \( A \) is irreflexive if \( a R a \) for every \( a \in A \).

Thus \( R \) is reflexive if every element \( a \in A \) is related to itself and it is irreflexive if no element is related to itself.

**Example 1**

(a) Let \( \Delta = \{(a, a) \mid a \in A\} \), so that \( \Delta \) is the relation of equality on the set \( A \). Then \( \Delta \) is reflexive, since \((a, a) \in \Delta \) for all \( a \in A \).

(b) Let \( R = \{(a, b) \in A \times A \mid a \neq b\} \), so that \( R \) is the relation of inequality on the set \( A \). Then \( R \) is irreflexive, since \((a, a) \not\in R \) for all \( a \in A \).

(c) Let \( A = \{1, 2, 3\} \), and let \( R = \{(1, 1), (1, 2)\} \). Then \( R \) is not reflexive since \((2, 2) \not\in R \) and \((3, 3) \not\in R \). Also, \( R \) is not irreflexive, since \((1, 1) \in R \).

(d) Let \( A \) be a nonempty set. Let \( R = \emptyset \subseteq A \times A \), the empty relation. Then \( R \) is not reflexive, since \((a, a) \not\in R \) for all \( a \in A \) (the empty set has no elements). However, \( R \) is irreflexive.

We can identify a reflexive or irreflexive relation by its matrix as follows. The matrix of a reflexive relation must have all 1's on its main diagonal, while the matrix of an irreflexive relation must have all 0's on its main diagonal.

Similarly, we can characterize the digraph of a reflexive or irreflexive relation as follows. A reflexive relation has a cycle of length 1 at every vertex, while an irreflexive relation has no cycles of length 1. Another useful way of saying this is that if \( \Delta \subseteq R \), then \( \Delta \) is reflexive if and only if \( \Delta \cap R = \Delta \).

Finally, we may note that if \( R \) is reflexive on a set \( A \), then \( \Dom(R) = \Ran(R) = A \).

- Symmetric, Antisymmetric, and Asymmetric Relations

A relation \( R \) on a set \( A \) is symmetric if whenever \( a R b \), then \( b R a \). It follows that \( R \) is not symmetric if we have some \( a \) and \( b \in A \) with \( a R b \) but \( b \not R a \). A relation \( R \) on a set \( A \) is asymmetric if whenever \( a R b \), then \( b \not R a \). It