4.5 Equivalence Relations

\{ \ldots, -5, -3, -1, 1, 3, 5, 7, \ldots \}, the set of odd integers, since each gives a remainder of 1 when divided by 2. Hence \( A/R \) consists of the set of even integers and the set of odd integers.

From Examples 7 and 8 we can extract a general procedure for determining partitions \( A/R \) for \( A \) finite or countable. The procedure is as follows:

**Step 1:** Choose any element of \( A \) and compute the equivalence class \( R(a) \).

**Step 2:** If \( R(a) \neq A \), choose an element \( b \), not included in \( R(a) \), and compute the equivalence class \( R(b) \).

**Step 3:** If \( A \) is not the union of previously computed equivalence classes, then choose an element \( x \) of \( A \) that is not in any of those equivalence classes and compute \( R(x) \).

**Step 4:** Repeat step 3 until all elements of \( A \) are included in the computed equivalence classes. If \( A \) is countable, this process could continue indefinitely. In that case, continue until a pattern emerges that allows you to describe or give a formula for all equivalence classes.

### 4.5 Exercises

In Exercises 1 and 2, let \( A = \{a, b, c\} \). Determine whether the relation \( R \) whose matrix \( M_A \) is given is an equivalence relation.

1. \[
M_A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

2. \[
M_A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

In Exercises 3 and 4 (Figures 4.26 and 4.27), determine whether the relation \( R \) whose digraph is given is an equivalence relation.

3. [Figure 4.26]

4. [Figure 4.27]

In Exercises 5 through 12, determine whether the relation \( R \) on the set \( A \) is an equivalence relation.

5. \( A = \{a, b, c, d\} \), \( R = \{(a, a), (b, a), (b, b), (c, c), (d, d), (d, c)\} \)

6. \( A = \{1, 2, 3, 4, 5\} \), \( R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2), (4, 4), (3, 3), (5, 5)\} \)

7. \( A = \{1, 2, 3, 4\} \), \( R = \{(1, 1), (1, 2), (1, 2), (2, 2), (2, 2), (3, 1), (3, 3), (4, 1), (4, 4)\} \)

8. \( A = \) the set of all members of the Software-of-the-Month Club; \( a \sim b \) if and only if \( a \) and \( b \) buy the same number of programs.

9. \( A = \) the set of all members of the Software-of-the-Month Club; \( a \sim b \) if and only if \( a \) and \( b \) buy the same programs.

10. \( A = \) the set of all people in the Social Security database; \( a \sim b \) if and only if \( a \) and \( b \) have the same last name.

11. \( A = \) the set of all triangles in the plane; \( a \sim b \) if and only if \( a \) is similar to \( b \).

12. \( A = \mathbb{Z}^+ \times \mathbb{Z}^+; (a, b) \sim (c, d) \) if and only if \( b = d \).

13. If \( \{a, b, c, d, f\} \) is a partition of the set \( A = \{a, b, c, d, e, f\} \), determine the corresponding equivalence relation \( R \).

14. If \( \{\{1, 3, 5\}, \{2, 4\}\} \) is a partition of the set \( A = \{1, 2, 3, 4, 5\} \), determine the corresponding equivalence relation \( R \).

15. If \( \{\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}\} \) is a partition of the set \( A = \{1, 2, 3, \ldots, 10\} \), determine the corresponding equivalence relation \( R \).

16. If \( \{\{a, i\}, \{e, a\}, \{u\}\} \) is a partition of the set \( A = \{a, e, i, a, u\} \), determine the corresponding equivalence relation \( R \).
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17. Let $A$ and $R$ be the set and relation defined in Example 5. Compute $A/R$.

18. Let $A = \{1, 2, 3, 4\}$ and $R$ be the relation on $A$ defined by

$$M_R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}$$

Compute $A/R$.

19. Let $A = \mathbb{R} \times \mathbb{R}$. Define the following relation $R$ on $A$: $(a, b), (c, d) \in R$ if and only if $a^2 + b^2 = c^2 + d^2$.

(a) Show that $R$ is an equivalence relation.

(b) Compute $A/R$.

20. Let $A = \{a, b, c, d, e\}$ and $R$ be the relation on $A$ defined by

$$M_R = \begin{bmatrix}
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

Compute $A/R$.

21. Let $S = \{1, 2, 3, 4, 5\}$ and $A = S \times S$. Define the following relation $R$ on $A$: $(a, b) R (a', b')$ if and only if $ab' = a'b$.

(a) Show that $R$ is an equivalence relation.

(b) Compute $A/R$.

22. Let $S = \{1, 2, 3, 4\}$ and $A = S \times S$. Define the following relation $R$ on $A$: $(a, b) R (a', b')$ if and only if $a + b = a' + b'$.

(a) Show that $R$ is an equivalence relation.

(b) Compute $A/R$.

23. A relation $R$ on a set $A$ is called circular if $a R b$ and $b R c$ imply $c R a$. Show that $R$ is reflexive and circular if and only if it is an equivalence relation.

24. Show that if $R_1$ and $R_2$ are equivalence relations on $A$, then $R_1 \cap R_2$ is an equivalence relation on $A$.

25. Define an equivalence relation $R$ on $Z$, the set of integers, different from that used in Examples 4 and 8 and whose corresponding partition contains exactly two infinite sets.

26. Define an equivalence relation $R$ on $Z$, the set of integers, whose corresponding partition contains exactly three infinite sets.

In Exercises 27 and 28, use the following definition. Given an equivalence relation $R$ on a set $A$, define the sum of $R$-related sets, $R(a) + R(b)$, by $\{ x \mid x = s + t, s \in R(a), t \in R(b) \}$.

27. Let $R$ be the equivalence relation in Example 4. Show that $R(a) + R(b) = R(a + b)$ for all $a, b$.

28. Let $R$ be the equivalence relation in Exercise 12. Show that $R(a) + R(b) = R(a + b)$ for all $a, b$.

29. Let $R$ be the equivalence relation in Exercise 21. Define $(a, b) + (a', b') = (a + a', b + b')$ for elements of $A$. Prove or disprove that $R((a, b)) + R((a', b')) = R((a + a', b + b'))$.

### 4.6 Computer Representation of Relations and Digraphs

The most straightforward method of storing data items is to place them in a linear list or array. This generally corresponds to putting consecutive data items in consecutively numbered storage locations in a computer memory. Figure 4.28 illustrates this method for five data items $D_1, \ldots, D_5$. The method is an efficient use of space and provides, at least at the level of most programming languages, random access to the data. Thus the linear array might be $A$ and the data would be in locations $A[1], A[2], A[3], A[4], A[5]$, and we would have access to any data item $D_i$ by simply supplying its index $i$.

The main problem with this storage method is that we cannot insert new data between existing data without moving a possibly large number of items. Thus, to add another item $E$ to the list in Figure 4.28 and place $E$ between $D_2$ and $D_3$, we would have to move $D_3$ to $A[4]$, $D_4$ to $A[5]$, and $D_5$ to $A[6]$, if room exists, and then assign $E$ to $A[3]$.

An alternative method of representing this sequence is by a linked list, shown in schematic fashion in Figure 4.29. The basic unit of information storage is the storage cell. We imagine such cells to have room for two information items. The first can be data (numbers or symbols), and the second item is a pointer, that is, a number that tells us (points to) the location of the next cell to be considered. Thus cells may be arranged sequentially, but the data items that they represent are not