Practice Midterm

Name:

Instructions - Read Carefully

1. To receive full credit solutions must be complete and clearly demonstrate the student’s reasoning.

2. Solutions must be organized in a logical fashion or students will not receive full credit.

3. No calculators, phones, or any other supplemental materials may be used on this exam.

4. Answers need not be simplified nor written in decimal form.

5. Any actions of academic dishonesty will result in a zero for the exam.

6. You have 50 minutes to complete this exam. Stop working immediately when time is called.
Total Score:
1) Answer an unambiguous True or False to each of the following:

(a) If $A \subset B$ then $|A| < |B|$.  

(b) For any sets $A, B$ there is at least one injective function $A \to B$.  

(c) If $f : A \to B$ is a bijection, then $|A| = |B|$.  

(d) The Cantor-Schröder-Bernstein Theorem says that every infinite set is in bijection with a proper subset of itself.  

(e) Every graph has an even number of vertices.  

(f) A graph with only two vertices of odd degree has an Eulerian path.  

(g) $\binom{1094}{594} = \binom{1094}{500}$  

(h) For finite sets, the number of injective functions $f : A \to B$ is $|B|^{|A|}$.  

(i) $\emptyset = \{\emptyset\}$  

(j) $0! = 0$  

(k) A bijection is an injective and surjective function.
2) For each of the following functions answer the following questions. Is it an injection? Is it a surjection? If the answer is no, then provide an example which demonstrates this.

(a) \( f : \mathbb{Z} \to \mathbb{Z}, \ m \mapsto 2m \)

(b) \( g : \mathbb{Q} \to \mathbb{Q}, \ q \mapsto 2q \)

(c) \( h : \{1, 2, 3\} \to \{1, 2, 3, 4\} \ n \mapsto n + 1 \)

(d) \( F : \mathbb{N} \to \mathbb{N}, \ n \mapsto n^2 \)

(e) \( G : \mathbb{Z} \to \mathbb{Z}, \ m \mapsto m^2 \)

(f) \( H : \{\varnothing, \wp, \sigma\} \to \{\wp, \wp, \wp, \wp\} \)
\( H(\varnothing) = \wp, H(\wp) = \wp, H(\sigma) = \sigma \)
3) Prove that the sets $A = \{ m \in \mathbb{Z} \mid m \text{ is odd and } m > 40 \}$ and $B = \{ m \in \mathbb{Z} \mid m > 1 \}$ have the same cardinality.
4) Is it possible to draw a graph whose vertices have the listed degrees? If so draw an example. If not explain what graph theoretic fact tells you this.

(a) 1,5
(b) 1,2,3,2
(c) 2,2,2
(d) 3,3,3,3
(e) 7,7,7,7,7,7,7,7,7
5) For which of the graphs is it possible to find an Eulerian circuit? For which an Eulerian path? Explain what fact about graph theory lets you know this. If an Eulerian circuit or path exists, demonstrate what it is.
6) Explain combinatorially why these identities hold.

(a) \( \binom{n}{1} = n \)

(b) \( \binom{n}{0} = 1 \)

(c) \( \binom{n}{n} = 1 \)

(d) \( \binom{n}{n-1} = n \)

(e) \( \binom{n}{k} = \binom{n}{n-k} \)
7) A very paranoid bank requires that its customers choose particularly secure and hard to remember passwords. How many possible passwords are there if:

(a) Passwords must be ten characters long and be from the set \{a, b, \ldots, y, z, A, B, \ldots, Y, Z\}.

(b) The same as the above, but each character must be unique.

(c) Each character must be unique, password is ten characters long, from the set \{0, 1, 2, \ldots, 9\}

(d) The same as (a), but it must start and end with a capital letter.
8) Prove the following identity using a double counting (combinatorial) argument. For \( n, k \in \mathbb{Z}_{\geq 0} \),

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]
9) Prove the following identity using a double counting (combinatorial) argument. For \( n, k \in \mathbb{Z}_{\geq 0}, k \leq n, \)

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]
10) Find the flaws in this faulty proof and only the actual flaws.

\textbf{Theorem} There is a bijection \( f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \).

\textit{Proof} To do so we give two examples of bijections.

\( f_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (n, n) \mapsto n + n \)

This is surjective since for \( b \in \mathbb{N} \) we have that \( \frac{b}{2} \in \mathbb{N} \) so that \( \left( \frac{b}{2}, \frac{b}{2} \right) \in \mathbb{N} \) and \( f_1 \left( \frac{b}{2}, \frac{b}{2} \right) = \frac{b}{2} + \frac{b}{2} = b \). Also notice it is injective since

\[
f_1(n_1, n_1) = f_1(n_2, n_2) \implies 2n_1 = 2n_2 \implies n_1 = n_2
\]

as desired. This shows this function is a bijection.

\( f_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad f(n, m) \mapsto n \cdot m \)

This is surjective since for \( b \in \mathbb{N} \) we may write \( b = b \cdot 1 \). So \( f_2(b, 1) = b \cdot 1 = b \) as desired. Next injectivity is shown. If \( f(n_1, m_1) = f(n_2, m_2) \), then \( n_1 \cdot m_1 = n_2 \cdot m_2 \) which implies \( n_1 = n_2 \) and \( m_1 = m_2 \) by prime factorization. Thus we are done.
11) Give an algebraic or a combinatorial proof of the following identity. Here $n, k \in \mathbb{Z}_{\geq 0}, k \leq n$.

\[
\binom{n}{2} \binom{n-2}{k-2} = \binom{n}{k} \binom{k}{2}
\]
Extra Credit Answer some very hard problem with your remaining time.