(1) Find the directional derivative of \( f \) at the point \( P \) in the direction of the vector \( \mathbf{a} \) where

(a) \( f(x, y) = e^x \cos y; \ P(0, \pi/4); \ \mathbf{a} = -3\mathbf{i} + 3\mathbf{j} \),
(b) \( f(x, y, z) = x^3z - yx^2 + z^2; \ P(2, -1, 1); \) and \( \mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

Comments: (a) \( \nabla f = (e^x \cos y, -e^x \sin y) \) so \( \nabla f(0, \pi/4) = (\sqrt{2}/2, -\sqrt{2}/2) \). Then the directional derivative at the point \( P \) is

\[
(D_{\mathbf{a}} f)(0, \pi/4) = \nabla f \cdot \frac{\mathbf{a}}{||\mathbf{a}||} = (\sqrt{2}/2, -\sqrt{2}/2) \cdot \frac{(-1, 1)}{\sqrt{2}} = -1.
\]

(b) We find that \( \nabla f = (3x^2z - xy, -x^2, x^3 + 2z) \) so \( \nabla f(P) = (14, -4, 10) \). Also \( ||\mathbf{a}|| = \sqrt{14} \). Then the directional derivative is given by

\[
(D_{\mathbf{a}} f)(P) = (14, -4, 10) \cdot \frac{(3, -1, 2)}{\sqrt{14}} = 38/\sqrt{14}.
\]

(2) Find the directional derivative of \( f(x, y) = x/(x+y) \) at \( P(1, 0) \) in the direction of the point \( Q(-1, -1) \).

Comments: \( \nabla f = \left( \frac{y}{(x+y)^2}, -\frac{x}{(x+y)^2} \right) \) so \( \nabla f(1, 0) = (0, -1) \). The vector \( \mathbf{a} = \overrightarrow{PQ} = (-2, -1) \) so

\[
(D_{\mathbf{a}} f)(1, 0) = (0, -1) \cdot \frac{-2, -1}{\sqrt{5}} = 1/\sqrt{5}.
\]

(3) Sketch the level curve of \( f(x, y) = x^2 + 4y^2 \) that passes through the point \( P(-2, 0) \) and draw the gradient vector at \( P \).

Comments: The level curve through \( P(-2, 0) \) is the ellipse \( x^2 + 4y^2 = 4 \) or \((x/2)^2 + y^2 = 1\).

(4) Find a unit vector in the direction in which \( f \) increases most rapidly at the point \( P \), and find the rate of change of \( f \) at \( P \) in that direction.

(a) \( f(x, y) = 3x - \ln y; \ P(-1, 1) \). (b) \( f(x, y) = x/(x+y); \ P(0, 2) \).

Comments: The direction is the direction of the gradient \( \nabla f \) and the maximal rate of change is \( ||\nabla f|| \).

(a) \( \nabla f = (3, -1/y) \) so \( \nabla f(-1, 1) = (3, -1) \) and \( ||\nabla f|| = \sqrt{10} \).

(b) \( \nabla f = \left( \frac{y}{(x+y)^2}, -\frac{x}{(x+y)^2} \right) \) so \( \nabla f(0, 2) = (1/2, 0) \) with \( ||\nabla f|| = 1/2 \).

(5) Let \( z = 3x^2 - y^2 \). Find all points at which \( ||\nabla z|| = 6 \).

Comments: \( \nabla f = (6x, -2y) \) so \( ||\nabla f|| = \sqrt{36x^2 + 4y^2} = 6 \) or \( 36x^2 + 4y^2 = 36 \). The standard form for this ellipse is \( x^2 + y^2/9 = 1 \).

(6) Given that the directional derivative of \( f(x, y, z) \) at the point \( P(3, -2, 1) \) in the direction of the vector \( \mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} \) is \(-5\) and that \( ||\nabla f(3, -2, 1)|| = 5 \), find \( \nabla f(3, -2, 1) \).

Comments: We know that \( (D_{\mathbf{a}} f)(P) = \nabla f(P) \cdot \frac{\mathbf{a}}{||\mathbf{a}||} = -5 \). Since \( ||\nabla f(3, -2, 1)|| = 5 \), we know that the minimal rate of change of \( f \) occurs in the direction of \( \mathbf{a} \) which is the direction of \(-\nabla f\). So \( \nabla f(P) = -5\mathbf{a}/||\mathbf{a}|| \).
(7) Consider the ellipsoid \( x^2 + y^2 + 4z^2 = 12. \) Find an equation of the tangent to the ellipsoid at the point \( P(2, 2, 1). \) Find parametric equations for its normal line at this point.

Comments: Let \( F(x, y, z) = x^2 + y^2 + 4z^2 = 12 \) so \( \nabla F = (2x, 2y, 8z) \). At the point \( P(2, 2, 1), \nabla F = (4, 4, 8) \). So \( \mathbf{N} = (1, 1, 2) \) is normal to the ellipsoid. Then the tangent plane is given by \( (x-2) + (y-2) + 2(z-1) = 0 \) and normal line \( x = 2 + t, y = 2 + t, z = 1 + 2t. \)

(8) Find all points on the ellipsoid \( 2x^2 + 3y^2 + 4z^2 = 9 \) at which the plane tangent to the ellipsoid is parallel to the plane \( x - 2y + 3z = 5. \)

Comments: Recall that two planes are parallel if and only if their normal vectors are multiples of each other. The normal to \( x - 2y + 3z = 5 \) is \( \mathbf{N} = (1, -2, 3). \) The normal vectors to a level surface \( F(x, y, z) = \) constant are given by the gradient \( \nabla F: \)

\[ \nabla F = \nabla(2x^2 + 3y^2 + 4z^2) = (4x, 6y, 8z). \]

We have the equation \( c\mathbf{N} = \nabla F \) or equivalently

\[ c(1, -2, 3) = (2x, 3y, 4z) \]

This single vector equation gives rise to 3 equations:

\( c = 2x, \quad -2c = 3y, \quad 3c = 4z; \)

that is,

\( x = c/2, \quad y = -2c/3, \quad z = 3c/4. \)

We substitute these equations back into the equation of the ellipsoid

\[ 2(c/2)^2 + 3(-2c/3)^2 + 4(3c/4)^2 = 9 \]

which reduces to \( (2/4 + 3 \cdot 4/9 + 9/4)c^2 = 9 \) or \( 49c^2/12 = 9; \) that is, \( 49c^2 = 108. \) Hence \( c = \pm 6\sqrt{3}/7. \)

We conclude that there are two points are the ellipsoid:

\( x = 3\sqrt{3}/7, \quad y = -4\sqrt{3}/7, \quad z = 9\sqrt{3}/14, \)

\( x = -3\sqrt{3}/7, \quad y = 4\sqrt{3}/7, \quad z = -9\sqrt{3}/14. \)

(9) Find all points on the surface \( x^2 + y^2 - z^2 = 1 \) at which the normal line is parallel to the line through \( P(1, -2, 1) \) and \( Q(4, 0, -1). \)

Comments: We want to find the normal line parallel to \( \overrightarrow{PQ} = \langle 3, 2, -2 \rangle. \) Let \( F(x, y, z) = x^2 + y^2 - z^2 \)
so \( \nabla F = \langle 2x, 2y, -2z \rangle. \) We require that \( \langle x, y, -z \rangle = c \langle 3, 2, -2 \rangle \) that gives the equations

\( x = 3c, \quad y = 2c, \quad -z = -2c; \)

We substitute these equations back into the equation for the hyperboloid to get

\( (3c)^2 + (2c)^2 - (2c)^2 = 1 \)

which reduces to \( 9c^2 = 1 \) so \( c = \pm 1/3. \) Hence we get two points: \((1, 2/3, 2/3)\) and \((-1, -2/3, -2/3).\)

(10) Find parametric equations for the tangent line to the curve of intersection of the paraboloid \( z = x^2 + y^2 \) and the ellipsoid \( x^2 + 4y^2 + z^2 = 9 \) at the point \( P(1, -1, 2). \)

Comments: A tangent vector to the curve of intersection is given by \( \mathbf{N}_1 \times \mathbf{N}_2 \) where \( \mathbf{N}_1 \) is normal to the graph of \( z = x^2 + y^2 \) at the point \( P \) and \( \mathbf{N}_2 \) is normal to the level surface \( F(x, y, z) = x^2 + 4y^2 + z^2 = 9 \) at the point \( P. \)

Now \( \mathbf{N}_1 = \langle -2x, -2y, 1 \rangle. \) At the point \( P, \mathbf{N}_1 = \langle -2, 2, 1 \rangle. \) The vector \( \mathbf{N}_2 = \nabla F = \langle 2x, 8y, 2z \rangle \).

At \( P, \mathbf{N}_2 = \langle 2, -8, 4 \rangle. \) We rescale and set \( \mathbf{N}_2 = \langle 1, -4, 2 \rangle. \)

We compute \( \mathbf{N}_1 \times \mathbf{N}_2 = \langle 8, 5, 6 \rangle \) which is tangent to the curve of intersection at the point \( P. \)

Hence the tangent line is given by \( x = 1 + 8t, y = -1 + 5t, z = 2 + 6t. \)

Max-Min Problems

(1) Find all critical points of \( f(x, y) = 4xy - x^4 - y^4 \) and classify them according to relative maxima, minima, and saddle points.

Comments: The partial derivatives of \( f(x, y) \) are

\[ \frac{\partial f}{\partial x} = 4y - 4x^3 = 0, \quad \frac{\partial f}{\partial y} = 4x - 4y^3 = 0. \]
Hence \( y - x^3 = 0 \) and \( x - y^3 = 0 \). So \( y = x^3 \). Substituting back, we get \( x - x^9 = 0 \) or \( x(1 - x^8) = 0 \). Hence \( x = 0 \) or \( x = \pm 1 \). The corresponding \( y \) values are \( y = 0 \), \( y = 1 \), and \( y = -1 \); that is, the critical points are

\[(0,0), \quad (1,1), \quad (-1,-1).\]

The second order partial derivatives are

\[
\frac{\partial^2 f}{\partial x^2} = -12x^2, \quad \frac{\partial^2 f}{\partial y^2} = -12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 4.
\]

The resulting \( 2 \times 2 \) determinant is

\[D = \begin{vmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{vmatrix} = 144x^2y^2 - 16.
\]

At \((0,0)\), \( D = -16 \) so \((0,0)\) is a saddle point; at \((1,1)\), \( D = 144 - 16 > 0 \). Since \( f_{xx}(1,1) = -12 < 0 \), \((1,1)\) is a relative maximum; at \((-1,-1)\), \( D \) is again positive and \( f_{xx}(-1,-1) \) is negative, so \((-1,-1)\) is a relative maximum.

(2) Find all critical points of \( f(x,y) = x^5 + y^5 - 5xy \). Classify them as a relative maximum, relative minimum, or saddle point.

**Comments:** We proceed as in the above problem. We find that \( f_x = 5x^4 - 5y = 0 \) so \( y = x^4 \) and \( f_y = 5y^4 - 5x = 0 \) so \( x = y^4 \). Combining, we now have \( x = (x^4)^4 = x^{16} \) so \( x^{16} - x = 0 \). Factoring, we obtain \( x(x^{15} - 1) = 0 \) with solutions \( x = 0 \) and \( x = 1 \) with the corresponding values of \( y \) as \( y = 0 \) and \( y = 1 \). In other words, the critical points are \((0,0)\) and \((1,1)\). Next, we compute the second order partials:

\[
f_{xx} = 20x^3, \quad f_{yy} = 20y^3, \quad f_{xy} = -5.
\]

Then the \( 2 \times 2 \) determinant \( D \) is

\[D = \begin{vmatrix} 20x^3 & -5 \\ -5 & 20y^3 \end{vmatrix} = 400x^3y^3 - 25.
\]

At \((0,0)\), \( D = -25 < 0 \) so \((0,0)\) is a saddle point.

at \((1,1)\), \( D = 400 - 25 > 0 \) and \( f_{xx}(1,1) > 0 \) so \((1,1)\) is a relative minimum.

(3) Find all points \((x,y,z)\) on the surface \( z^2 - xy = 5 \) that are closest to the origin.

**Comments:** We need to minimize the distance from a point on the surface to the point \((0,0,0)\); that is, \(\sqrt{x^2 + y^2 + z^2} \). Almost always, it is easier to minimize the distance squared: \(x^2 + y^2 + z^2 \). Since \( z^2 = 5 + xy \) on the surface, this becomes

\[f(x,y) = x^2 + y^2 + xy.
\]

We need to find the critical points of \( f \):

\[
\frac{\partial f}{\partial x} = 2x + y = 0, \quad \frac{\partial f}{\partial y} = 2y + x = 0.
\]

We obtain the linear system: \(2x + y = 0\) and \(x + 2y = 0\) which has the only solution \((0,0)\) which is the critical point of \( f \). Further, the second order partial derivatives of \( f \) are

\[
f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1.
\]

Hence,

\[D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.
\]

Since \( f_{xx} > 0 \), \((0,0)\) is a minimum. Since \( z^2 = 5 + xy = 5 \), the corresponding \( z \)-values are \(\pm \sqrt{5} \). So there are two points on the surface at minimum distance \((0,0,\pm \sqrt{5}) \).

(4) Find the absolute extrema of \( z = f(x,y) = x^2 + y^2 \) on the domain \( D \) bounded by the ellipse \( x^2/4 + y^2/9 = 1 \).

**Comments:** This is a two-part problem. First, we need to find all the relative extrema inside the ellipse; then, second, we need to check the boundary ellipse itself for extrema.

The critical points of \( f(x,y) = x^2 + y^2 \) are given by \( f_x = 2x = 0 \) and \( f_y = 2y = 0 \). So there is only one critical point \((0,0)\) with \( f(0,0) = 0 \). Note that \( D = 4 \) and \( f_{xx} = 2 \) so it is a minimum and
not a saddle point. 
On the boundary ellipse, \( x = 2 \cos t, y = 3 \sin t \) with \( 0 \leq t \leq 2\pi \). Then \( z = (2 \cos t)^2 + (3 \sin t)^2 = 4 \cos^2 t + 9 \sin^2 t \). We need to find its absolute extrema. Now
\[
\frac{dz}{dt} = 10 \cos t \sin t,
\]
so its critical points are \( 0, \pi/2, \pi, 3\pi/2 \). But \( z = z(t) = z(0) = 4, z(\pi/2) = 9, z(\pi) = 4, \) and \( z(3\pi/2) = 9 \). Hence, the absolute maximum of \( f \) on \( D \) is 9 while the absolute minimum of \( f \) on \( D \) is 0.

(5) A rectangular box without a lid is made from 12 square meters of cardboard. Find the maximum volume such a box.

Comments: Denote the length, width, and height of the box by \( x, y, \) and \( z \), respectively. So its volume is \( V = xyz \). We can solve for \( z \) in terms of \( x, y \) since its surface area is 12; that is, \( 2xz + 2yz + xy = 12 \) shows \( z = (12 - xy)/[2(x + y)] \). Hence,
\[
V = V(x, y) = xy \frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}.
\]
We now find its critical points where both partial derivatives are zero:
\[
\frac{\partial V}{\partial x} = 1/2 \frac{y^2 (12 - x^2 - 2xy)}{(x + y)^2}, \quad \frac{\partial V}{\partial y} = 1/2 \frac{x^2 (12 - x^2 - 2xy)}{(x + y)^2}.
\]
These derivative are both zero if \( x = y = 0 \). This solution is not physical. So, we examine the two equations:
\[
12 - x^2 - 2xy = 0, \quad 12 - y^2 - 2xy = 0.
\]
Hence, \( x^2 = y^2 \) or \( x = y \) since \( x, y \geq 0 \). Substitute this condition in \( 12 - x^2 - 2xy = 0 \) to obtain \( 12 - 3x^2 \) so \( x = 2 \). Similarly, \( y = 2 \). Then \( z = (12 - xy)/[2(x + y)] = 1 \). We conclude that the maximal volume \( V(x, y) = xyz = 4 \).

(6) Find the absolute maximum and minimum values of the function \( f(x, y) = x^2 - 2xy + 2y \) on the rectangle \( D = \{ (x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2 \} \).

Comments: Since \( D \) is a closed and bounded set, the function \( f \) will have absolute extrema. First we find the critical points where both partial derivatives are zero:
\[
\frac{\partial f}{\partial x} = 2x - 2y = 0, \quad \frac{\partial f}{\partial y} = -2x + 2 = 0.
\]
There is only one critical point \((1, 1)\) and the value of \( f \) there is \( f(1, 1) = 1 \).

Next we need to find the extrema on each of the boundary lines, say \( L_1, L_2, L_3, \) and \( L_4 \) where \( L_1 \) is on the \( x \)-axis \((y = 0)\) with \( 0 \leq x \leq 3 \); \( L_2 \), the vertical line \( x = 3 \), with \( 0 \leq y \leq 2 \); the horizontal line \( L_3 \), with \( y = 2 \) and \( 0 \leq x \leq 3 \); and \( L_4 \), the vertical line \( x = 0 \) and \( 0 \leq y \leq 2 \).

On \( L_1 \), \( f(x, 0) = x^2 \). This function is increasing. Its minimum value is \( f(0, 0) = 0 \) and maximum value is \( f(3, 0) = 9 \).

On \( L_2 \), \( f(3, y) = 9 - 4y \). This function is decreasing. Its maximum value is \( f(3, 0) = 9 \) and minimum value is \( f(3, 2) = 1 \).

On \( L_3 \), \( f(x, 2) = x^2 - 4x + 4 = (x - 2)^2 \). Its minimum value is \( f(2, 2) = 0 \) and maximum value is \( f(0, 4) = 4 \) since \( 0 \leq x \leq 3 \).

On \( L_4 \), \( f(0, y) = 2y \). Its maximum value is \( f(0, 2) = 4 \) and minimum value is \( f(0, 0) = 0 \).

Conclude: on the edges, the minimum value of \( f \) is 0 while its maximum value is 9.

Since the value of \( f \) at the interior critical point is 1, the absolute extrema occur on the boundary.