Problem set 2

1. A line segment of unit length is cut once at random. What is the probability that the longer piece is more than twice the length of the shorter piece?

2. Choose a number $X$ at random from the interval $[0, 1]$. Find the probability that
   (a) $1/3 < X < 2/3$
   (b) $|X - 1/2| \leq 1/4$
   (c) $X < 1/4$ or $1 - X < 1/4$
   (d) $3X^2 < X$

3. Choose a number $X$ at random from the interval $[2, 10]$. Find the probability that
   (a) $X > 5$
   (b) $5 < X < 7$
   (c) $X^2 - 12X + 35 > 0$

4. Choose a point $(X, Y)$ at random in the unit square $[0, 1] \times [0, 1]$. Find the probability that
   (a) $X + Y < 1/2$
   (b) $XY < 1/2$
   (c) $|X - Y| < 1/2$
   (d) $\max(X, Y) < 1/2$
   (e) $\min(X, Y) < 1/2$
   (f) $X < 1/2$ and $1 - Y < 1/2$
   (g) conditions (c) and (f) both hold
   (h) $X^2 + Y^2 < 1/2$

5. Let $X$ be a continuous random variable whose density function is $f(x) = cx(1-x)$ for $0 < x < 1$, and $f(x) = 0$ otherwise. Find the value of $c$, $P(X \leq 1/2)$, $P(X \leq 1/3)$, and $P(1/3 \leq X \leq 1/2)$.

6. Let $X$ be a continuous random variable uniformly distributed in $[-1, 1]$. Find the cumulative distribution function and the density function of $|X|$.

7. A radioactive source emits particles at a rate described by the exponential density $f(t) = e^{-t}$, so that the probability that a particle will appear in the next $t$ seconds is $P([0, t]) = \int_0^t e^{-s} ds$.
   Find the probability that a particle (not necessarily the first) will appear
   (a) within the next second
   (b) within the next 3 seconds
   (c) between 3 and 4 seconds from now
   (d) after 4 seconds from now

8. Assume that a new light bulb will burn out after $T$ hours, where the density of $T$ is $f(t) = \lambda e^{-\lambda t}$. Find the probability that the bulb will not burn out before $t$ hours. For what $t$ is this probability $1/2$?

9. Each of the two numbers, $B$ and $C$, is chosen at random from the interval $[-1, 1]$. Find the probability that both roots of the quadratic equation $x^2 + Bx + C = 0$ are real.

10. A stick of unit length is broken into three pieces. The break points are chosen simultaneously and at random. Find the probability that the three pieces can be used to form a triangle.