Notes on LU

An LU factorization of an \( m \times n \) matrix is its expression in the form of a product of an \( m \times m \) unit lower triangular matrix \( L \) and an \( m \times n \) upper triangular matrix \( U \).

**EXAMPLE**

\[
\begin{bmatrix}
1 & 2 \\
3 & 5
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}
\]

If \( A = LU \), then \( U \) is a row *echelon* form of \( A \) obtained through the succession of elementary row operations

\[
\text{[ row } i \text{ of } A \text{ ] } - L_{ij} \text{ [ row } j \text{ of } A \text{ ]}, \quad i = j + 1, \ldots, m,
\]

for each \( j = 1, \ldots, m - 1 \). Here \( L_{ij} \) are the entries of \( L \) (below the diagonal).

The factor \( L \) is always invertible; its inverse is again a unit lower triangular matrix.

The factor \( U \) is invertible if and only if the matrix \( A \) itself is invertible. If it is, its inverse is also upper triangular.

**EXAMPLE**

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

Given \( A = LU \), we may deal with a linear system \( Ax = b \) in two takes: solve for \( Ux \) by forward substitution and then solve (if possible) for \( x \) by backward substitution. This approach is computationally efficient.

Not every matrix can be row reduced without row interchanges and so not every matrix has an LU factorization. To go around this problem, one permutes the rows of a matrix (pivoting operation). For instance, it is not possible to write \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) in the form \( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \) : the relations \( ax = 0, ay = 1, bx = 1 \) are inconsistent. However, \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) presents no difficulty.

If \( A \) does not have an LU factorization, but \( P \) is a permutation matrix such that \( PA \) has one, \( PA = LU \), then \( A \) can be written in the form \( A = P^{-1}LU = P^{T}LU \), which has many of the same virtues.

**EXAMPLE**

\[
\begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}
\]

The following example shows that more than one LU factorization may be possible.

**EXAMPLE**

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
=\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & x & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \text{ correct for any } x.
\]

This is not the case for invertible matrices. Indeed, if \( A = LU = \tilde{L}\tilde{U} \) is invertible, then

\[
\tilde{L}^{-1}L = \tilde{U}U^{-1}.
\]

Since \( \tilde{L}^{-1}L \) is unit lower triangular and \( \tilde{U}U^{-1} \) is upper triangular, the preceding equality,

\[
\begin{bmatrix}
1 & 0 & 0 \\
* & 1 & 0 \\
* & * & 1
\end{bmatrix}
=\begin{bmatrix}
* & * & * \\
0 & * & * \\
0 & 0 & *
\end{bmatrix},
\]

can hold only if the matrix on each side is the identity. Hence \( L = \tilde{L} \) and \( U = \tilde{U} \).

So both \( L \) and \( U \) are uniquely determined.

It is sometimes possible to make \( U \) unit upper triangular at the expense of inserting a diagonal matrix factor in the middle.

**EXAMPLE**

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
=\begin{bmatrix}
1 & 0 & 0 \\
4 & 1 & 0 \\
7 & 2 & 1
\end{bmatrix}\begin{bmatrix}
1 & 2 & 3 \\
0 & -3 & -6 \\
0 & 0 & 0
\end{bmatrix}
=\begin{bmatrix}
1 & 0 & 0 \\
4 & 1 & 0 \\
7 & 2 & 1
\end{bmatrix}\begin{bmatrix}
1 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & 0
\end{bmatrix}\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

This is an LDU factorization. The pivot entries are grouped on the diagonal of \( D \).

When \( A \) is symmetric, this is reflected in its LDU factorization.

**EXAMPLE**

\[
\begin{bmatrix}
2 & 4 \\
4 & 5
\end{bmatrix}
=\begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}\begin{bmatrix}
2 & 4 \\
0 & -3
\end{bmatrix}
=\begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}\begin{bmatrix}
2 & 0 \\
0 & -3
\end{bmatrix}\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\]

In this case, \( L = U^T \) and so \( A = LDL^T \), highlighting the symmetry.

When \( A \) is invertible, LDU leads to a quick factorization of the inverse: \( A^{-1} = U^{-1}D^{-1}L^{-1} \).

**EXAMPLE**

\[
\begin{bmatrix}
2 & 4 \\
4 & 5
\end{bmatrix}^{-1}
=\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}^{-1}\begin{bmatrix}
2 & 0 \\
0 & -3
\end{bmatrix}^{-1}\begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}^{-1}
=\begin{bmatrix}
1 & -2 \\
0 & 1
\end{bmatrix}\begin{bmatrix}
1/2 & 0 \\
0 & -1/3
\end{bmatrix}\begin{bmatrix}
1 & 0 \\
-2 & 1
\end{bmatrix}
\]