RECONSTRUCTING A SYSTEM FROM ITS SOLUTION

Example 1. Find a system of linear equations whose solution set is given by \( \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \).

**Answer.** Let \( x_1 \) and \( x_2 \) be the unknowns. Since the solution set consists of all possible multiples of \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \), the zero vector \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) is a solution. So the system must be homogeneous. How many equations are needed? Clearly, we can get away with one equation:

\[
\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0
\]
or

\[
2x_1 - x_2 = 0.
\]

The solution set of this equation is exactly as prescribed. In fact, any number of equations is possible: choose any \( n \times 2 \) matrix such that its rows are multiples of \( \begin{bmatrix} 2 & -1 \end{bmatrix} \) and at least one of the rows is nonzero. Then the resulting system in two unknowns has one basic variable and one free variable and its augmented matrix is row equivalent to

\[
\begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.
\]

For instance,

\[
\begin{align*}
-2x_1 + x_2 &= 0 \\
x_1 - 0.5x_2 &= 0
\end{align*}
\]

is just as good of an answer.

Example 2. Find a linear system whose solution set is given by \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \).

**Answer.** This time the system is nonhomogeneous: the zero vector \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) does not belong to the solution set of vectors \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} t - 1 \\ 2t + 1 \end{bmatrix} \), because \( t \) cannot equal 1 and \(-1/2\) at the same time. Observe that the solution set has “particular + homogeneous” form and that we already know how to deal with the homogeneous part. It remains to find \( b \) such that

\[
\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b
\]

has the solution set as prescribed. Clearly,

\[
b = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3.
\]

Thus \( 2x_1 - x_2 = -3 \) is a possible answer.
Example 3. Find a linear system whose solution set is given by \[
\begin{bmatrix}
-1 \\
1 \\
3 
\end{bmatrix} + \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.
\]

Answer. The solution set does not contain the zero vector, so the system must be nonhomogeneous. How many equations are needed? Since there are three variables and only one of them is free, one equation is insufficient to create two basic variables. We would need two equations. Consider the homogeneous part first and write the reduced augmented matrix:
\[
\begin{bmatrix}
1 & 0 & \ast & 0 \\
0 & 1 & \ast & 0 
\end{bmatrix}.
\]
How to choose \(\ast\)'s? Of course, for \[
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\] to be a solution, \(\ast = -1\).

The solution set of the augmented matrix
\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 
\end{bmatrix}
\]
consists of all multiples of \[
\begin{bmatrix} 1 \\ 1 
\end{bmatrix}.
\]
Now, for the nonhomogeneous part, we need to determine \(b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\) such that
\[
\begin{bmatrix}
-1 \\
1 \\
3 
\end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 
\end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 
\end{bmatrix}.
\]
Putting things together, we obtain an answer:
\[
\begin{align*}
x_1 - x_3 &= -4 \\
x_2 - x_3 &= -2
\end{align*}
\]

Example 4. Find a linear system whose solution set is given by \[
\begin{bmatrix}
-1 \\
1 \\
3 
\end{bmatrix} + \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}.
\]

Answer. Let’s use another method. A typical solution vector is of the form
\[
x = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},
\]
for some scalars \(s\) and \(t\). Thus
\[
\begin{align*}
x_1 &= -1 + s + 2t \\
x_2 &= 1 + s - 2t \\
x_3 &= 3 + s + t.
\end{align*}
\]
Now eliminate \(s\) and \(t\). Observe that \((x_1 + 1) - (x_2 - 1) = 4t\) and \((x_1 + 1) - (x_3 - 3) = t\). So
\[
x_1 - x_2 + 2 = 4(x_1 - x_3 + 4)
\]
or
\[
3x_1 + x_2 - 4x_3 = -14.
\]
The preceding equation has the solution set as prescribed. Indeed, both \(x_1 = x_2 = x_3 = 1\) and \(x_1 = 2, x_2 = -2, x_3 = 1\) satisfy the homogeneous equation \(3x_1 + x_2 - 4x_3 = 0\) and \(x_1 = -1, x_2 = 1, x_3 = 3\) is a particular solution of \(3x_1 + x_2 - 4x_3 = -14\).