**Bernoulli's theorem**

Below is a (modern form of a) result of Jacob Bernoulli (1655–1705).

It is an instance of the law of large numbers.

Let a sequence of independent trials be performed and let $p$ be the chance of an outcome $A$ in each trial. Let $\hat{m} = \hat{m}(n)$ be the number of times that $A$ has occurred in $n$ trials.

Then for any positive $\varepsilon$, no matter how small,

$$\left| \frac{\hat{m}}{n} - p \right| < \varepsilon$$

approaches 1, as $n$ increases without bound.

**PROOF**

Let $X_i$ be a 1/0 indicator of whether $A$ has occurred in the $i$-th trial.

Then $X_1 + \ldots + X_n = \hat{m}$.

We have $\mathbb{E}[X_i] = p$ and $\text{Var}(X_i) = p - p^2$.

So $\mathbb{E}[\hat{m}/n] = p$ and $\text{Var}(\hat{m}/n) = \frac{p(1 - p)}{n}$.

Fix any $\varepsilon > 0$. By Chebyshev's inequality,

$$P\left( \left| \frac{\hat{m}}{n} - p \right| < \varepsilon \right) \geq 1 - \frac{p(1 - p)}{n \varepsilon^2}.$$ 

But the right side of the inequality evidently tends to 1, as $n \to \infty$. $\square$

Thus, after sufficiently many trials, the observed relative frequency of the occurrence of an outcome will, with a high degree of certainty, stay within a given $\varepsilon$ of the outcome probability.

As the number of trials increases without bound, the degree of certainty increases to 1.

One says that, as $n \to \infty$, the relative frequency $\hat{m}/n$ converges to $p$ in probability.