Problems set 1.1

1. Solve \[ \begin{align*}
4x + 3y &= 2 \\
7x + 5y &= 3
\end{align*} \] using elimination. Check your answer.

2. Solve \[ \begin{align*}
x + 2y + 3z &= 8 \\
x + 3y + 3z &= 10 \\
x + 2y + 4z &= 9
\end{align*} \] using elimination. Check your answer.

3. The sums of any two of three numbers are 24, 28, and 30. Find these three numbers.

4. Find a polynomial of the form \( a + bt + ct^2 \) whose graph goes through the points \((1, -1), (2, 3), (3, 13)\).

5. Find an ellipse in the form \( ax^2 + bxy + cy^2 = 1 \) that runs through the points \((1, 2), (2, 2), (3, 1)\).

6. Solve the lower triangular system
\[ \begin{align*}
x_1 &= -3 \\
-3x_1 + x_2 &= 14 \\
x_1 + 2x_2 + x_3 &= 9 \\
-x_1 + 8x_2 - 5x_3 + x_4 &= 33.
\end{align*} \]

7. Find a system of linear equations in three unknowns whose solution set is the line through \((1, 1, 1)\) and \((3, 5, 0)\).

8. I have 32 bills in my wallet, in the denominations of $1, $5, and $10, worth $100 in total. How many do I have of each denomination?

Problems set 1.2

1. Solve \[ \begin{align*}
x + y - 2z &= 5 \\
2x + 3y + 4z &= 2
\end{align*} \] using Gaussian elimination. Show all work.

2. Solve \[ \begin{align*}
x_1 - 7x_2 + x_5 &= 3 \\
x_3 - 2x_5 &= 2 \\
x_4 + x_5 &= 1
\end{align*} \] using Gaussian elimination. Show all work.

3. Solve \[ \begin{align*}
4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\
5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\
-2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\
11x_1 + 6x_2 + 4x_3 + x_4 &= 11
\end{align*} \] using Gaussian elimination. Show all work.

4. For which values of \( a, b, c, d, e \) is \( \begin{bmatrix} 0 & a & 2 & 1 & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & e & 0 \end{bmatrix} \) in reduced row-echelon form?

5. How many types of \( 2 \times 3 \) matrices in reduced row-echelon form are there?
6. Is there a sequence of elementary row operations that transforms \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\] into \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]? Explain.

7. Consider the chemical reaction \( a\text{NO}_2 + b\text{H}_2\text{O} \rightarrow c\text{HNO}_2 + d\text{HNO}_3 \), where \( a, b, c, d \) are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. Balance this reaction using \( a, b, c, d \) as small as possible.

8. Consider a triangular plate with vertices at \( \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \). How should a total mass of 1 kg be distributed among the three vertices so that the plate can be supported at the point \( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \)? Assume that the mass of the plate itself is negligible.

**Problem set 1.3**

1. How many solutions does a linear system have if the reduced row-echelon form of its augmented matrix is \[
\begin{bmatrix}
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]?

2. Find the rank of \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\].

3. Consider a system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like?

4. Consider a system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like?

5. If the rank of a 5 \( \times \) 3 matrix \( A \) is 3, what is \( \text{rref}(A) \)?

6. Is the vector \[ \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \] a linear combination of \[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \] and \[ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \]?

7. For which values of \( b \) and \( c \) is \[ \begin{bmatrix} 3 \\ b \\ c \end{bmatrix} \] a linear combination of \[ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix} \]?

8. Consider an \( n \times m \) matrix \( A \) with more rows than columns. Argue that there is an \( n \times 1 \) vector \( b \) such the system with augmented matrix \[ \begin{bmatrix} A \\ b \end{bmatrix} \] is inconsistent.