Consider a deck of \( d \) cards numbered \( 1, \ldots, d \), with each card being equally likely to be drawn.

The expected value on a randomly selected card is then \( \frac{1 + \ldots + d}{d} = \frac{d+1}{2} \); the variance is
\[
\frac{1}{d} \sum_{k=1}^{d} k^2 - \frac{(d+1)^2}{4} = \frac{(d+1)(2d+1)}{6} - \frac{(d+1)^2}{4} = \frac{d+1}{2} \left( \frac{2d+1}{3} - \frac{d+1}{2} \right) = \frac{d^2 - 1}{12}.
\]

Let the experiment of drawing a random card (with replacement) be repeated over and over again.

After a large number of draws \( n \), how probable is it that a particular number \( k \) appears at least \( m_1 \) and at most \( m_2 \) times? A usual argument yields an exact answer,
\[
\sum_{m=m_1}^{m_2} \binom{n}{m} \left( \frac{1}{d} \right)^m \left( 1 - \frac{1}{d} \right)^{n-m},
\]
but evaluation of such a binomial sum is a computational challenge. So one invokes a limit theorem.

View the number of times \( M = M(k, n) \) that a particular number \( k \) appears in \( n \) trials as a sum of binary indicators \( X_i = \begin{cases} 1, & \text{ith trial yields } k \\ 0, & \text{otherwise} \end{cases} \). Also write \( p = \frac{1}{d} \) and \( q = 1 - \frac{1}{d} \).

The random variables \( X_i = X_i^2 \) are independent and identically distributed,
\[
E[X_i] = 1 \cdot \frac{1}{d} + 0 \cdot \left( 1 - \frac{1}{d} \right) = \frac{1}{d} = p \\
Var(X_i) = \left( 1 - \frac{1}{d} \right)^2 \cdot \frac{1}{d} + \left( 0 - \frac{1}{d} \right)^2 \cdot \left( 1 - \frac{1}{d} \right) = \frac{1}{d} \left( 1 - \frac{1}{d} \right) = pq.
\]

So their sum admits a simple analysis,
\[
E[M] = E[X_1 + \ldots + X_n] = E[X_1] + \ldots + E[X_n] = np \\
Var(M) = Var(X_1 + \ldots + X_n) = Var(X_1) + \ldots + Var(X_n) = npq.
\]

For sufficiently large \( n \), the normalized random variable \( \frac{M - np}{\sqrt{npq}} \) is nearly standard normal.

This was proved by de Moivre (1730) for \( p = 1/2 \), and by Laplace (1812) for a general \( 0 < p < 1 \).

In fact, one distinguishes between the local
\[
P(M = m) \approx \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}} e^{-(m-np)^2/2npq}
\]
and the integral
\[
P(m_1 \leq M \leq m_2) \approx \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-x^2/2} dx, \quad x_1 = \frac{m_1 - np}{\sqrt{npq}}, \quad x_2 = \frac{m_2 - np}{\sqrt{npq}}
\]
approximation forms of the de Moivre–Laplace limit theorem. Each is a consequence of the other.

Similarly, we may address the likelihood that the sum of \( n \) numbers drawn falls within certain bounds.

If \( Y_i \) is the \( i \)th outcome and \( S = Y_1 + \ldots + Y_n \), then \( E[S] = n(d+1)/2 \) and \( \text{Var}(S) = n(d^2 - 1)/12 \).

For large \( n \) then, \( S \) is close to \( \mathcal{N}(n(d+1)/2, n(d^2 - 1)/12) \) in distribution. This relies on a more general form of the central limit theorem, see Grinstead & Snell, page 343.