Euler’s Gamma Function

For $x > 0$, the integral $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ defines a positive convex function, differentiable to all orders.

It is easy to see that $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$.

For $x > 0$, integration by parts gives the functional equation $\Gamma(x + 1) = x \Gamma(x)$:

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt$$
$$= \int_0^\infty t^x d(-e^{-t})$$
$$= -t^x e^{-t}\bigg|_{t=0}^{t=\infty} + \int_0^\infty e^{-t}d(t^x)$$
$$= 0 + x \int_0^\infty t^{x-1} e^{-t} dt$$
$$= x \Gamma(x).$$

The functional equation implies that for any positive integer $n$, $\Gamma(n + 1) = n!$.

It also implies that $\lim_{x \to 0^+} x \Gamma(x) = \lim_{x \to 0^+} \Gamma(1 + x) = \Gamma(1) = 1$, and so

$$\Gamma(x) \sim \frac{1}{x}, \quad x \to 0^+. $$

At infinity $\Gamma(x)$ grows extremely rapidly: $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$, $x \to \infty$.

This is known as Stirling’s formula.