Inclusion-exclusion

Suppose that $A$, $B$, and $C$ are events (of a finite sample space).

The following equality holds:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$$- P(A \cap B) - P(B \cap C) - P(A \cap C)$$

$$+ P(A \cap B \cap C).$$

For a proof, consider any outcome $\omega$ in the union $A \cup B \cup C$.

Let $p = m(\omega)$ be the probability mass of $\omega$.

Then the contribution of $\omega$ to $P(A \cup B \cup C)$ is $p$.

To check that $\omega$ contributes as much to the right side,

examine three mutually exclusive possibilities:

$\omega$ belongs to exactly one of the sets: $(p + 0 + 0) - (0 + 0 + 0) + (0) = p$

$\omega$ belongs to exactly two of the sets: $(p + p + 0) - (p + 0 + 0) + (0) = p$

$\omega$ belongs to all three of the sets: $(p + p + p) - (p + p + p) + (p) = p$.

It follows that

$$\sum_{\omega \text{ in } A \cup B \cup C} m(\omega) = \sum_{\omega \text{ in } A} m(\omega) + \sum_{\omega \text{ in } B} m(\omega) + \sum_{\omega \text{ in } C} m(\omega)$$

$$- \sum_{\omega \text{ in } A \cap B} m(\omega) - \sum_{\omega \text{ in } B \cap C} m(\omega) - \sum_{\omega \text{ in } A \cap C} m(\omega)$$

$$+ \sum_{\omega \text{ in } A \cap B \cap C} m(\omega),$$

as desired.

The above identity can be generalized to any number of events $A_1, \ldots, A_n$:

$$P \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i} P(A_i) - \sum_{i,j} P(A_i \cap A_j) + \sum_{i,j,k} P(A_i \cap A_j \cap A_k) - \ldots .$$

To see this, note that every outcome $\omega$ contributes equally to both sides of the equation.

Indeed, if $\omega$ belongs to $r = 1, 2, \ldots, n$ of the sets $A_i$, it contributes $m(\omega)$ to the left side and

$$\left[ \binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \ldots + (-1)^{r-1} \binom{r}{r} \right] m(\omega) = m(\omega)$$

to the right side.