A property of the mean

Let \( a_1, \ldots, a_n \) be arbitrary real numbers and let \( w_1, \ldots, w_n \) be positive weights summing to 1. Then, for any \( x \) on the real line, the identity

\[
\sum_{j=1}^{n} w_j (a_j - x)^2 = \sum_{j=1}^{n} w_j (a_j - \bar{a})^2 + (x - \bar{a})^2,
\]

holds with \( \bar{a} = \sum_{j=1}^{n} w_j a_j \). In other words, the minimum value of the quadratic \( q(x) = \sum_{j=1}^{n} w_j (a_j - x)^2 \) is attained at \( x = \bar{a} \) and equals \( q(\bar{a}) = \sum_{j=1}^{n} w_j (a_j - \bar{a})^2 \).

If \( A \) is a random variable taking on the values \( a_j \) with respective probabilities \( w_j \), then \( E[A] = \bar{a} \) is the expected value of \( A \), \( q(x) = E[(A - x)^2] \) is the mean squared distance to \( x \), and the minimum value \( q(\bar{a}) = \text{Var}(A) \) is the variance of \( A \).

In this context, the identity takes the form

\[
E[(A - x)^2] = \text{Var}(A) + (x - \bar{a})^2.
\]

If we view \( (a_j, w_j), j = 1, \ldots, n, \) as a system of point masses on the real line, then \( J_x = \sum_{j=1}^{n} w_j (a_j - x)^2 \) represents the moment of inertia with respect to a given point \( x \). The fact that the moment of inertia is minimized when \( x \) is the centroid of the system \( \bar{a} \), as well as the identity

\[
J_x = J_{\bar{a}} + |x - \bar{a}|^2,
\]

goes back to Lagrange.

Setting \( x = 0 \), we obtain the following consequence:

\[
\sum_{j=1}^{n} w_j a_j^2 \geq \left( \sum_{j=1}^{n} w_j a_j \right)^2,
\]

with equality if and only if \( \sum_{j=1}^{n} w_j (a_j - \bar{a})^2 = 0 \), i.e., all \( a_j \) are the same. This is an instance of the Cauchy–Schwarz inequality.

One way to prove the identity, is to appropriately rearrange the terms. More precisely, using probabilistic notation, we have

\[
E[(A - x)^2] = E[(A - \bar{a} + \bar{a} - x)^2]
= E[(A - \bar{a})^2] + 2(\bar{a} - x)E[A - \bar{a}] + (\bar{a} - x)^2
= \text{Var}(A) + 0 + (\bar{a} - x)^2.
\]