1.6 Applications of Linear Systems

Many everyday problems are set up as linear systems.

Possible solution sets:

1) no solutions
2) one unique solution
3) an infinite number of solutions

Applications:

1) a homogeneous system in economics
2) balancing chemical equations
3) network flow

A Homogeneous System in Economics

Leontief’s “exchange model”:

1) economy sectors: coal, electric, steel
2) known: output from each sector to each sector
3) unknown: total output prices for each sector
4) “equilibrium prices”: as output prices, such that incomes balance expenses

<table>
<thead>
<tr>
<th>Distribution of Output from:</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Coal</td>
<td>Electric</td>
<td>Steel</td>
<td>Purchased by:</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>.4</td>
<td>.6</td>
<td>Coal</td>
<td></td>
</tr>
<tr>
<td>.6</td>
<td>.1</td>
<td>.2</td>
<td>Electric</td>
<td></td>
</tr>
<tr>
<td>.4</td>
<td>.5</td>
<td>.2</td>
<td>Steel</td>
<td></td>
</tr>
</tbody>
</table>

Unknowns: find (if possible) \( p_C, p_E, p_S \) \( \in \) income match expenditures

Solution:

Columns represent sector’s output (totals to 100%)

Rows represent sector’s input (expenditures)

Setting up equations for equilibrium (i.e., balance).

For Coal:
\[ p_C = .4p_E + .6p_S \]

For Electricity:
\[ p_E = .6p_C + .1p_E + .2p_S \]

For Steel:
\[ p_S = .4p_C + .5p_E + .2p_S \]
Rearranging we arrive at *homogeneous* system:

\[
p_C - .4 p_E - .6 p_S = 0 \\
- .6 p_C + .9 p_E - .2 p_S = 0 \\
- .4 p_C - .5 p_E + .8 p_S = 0
\]

Row reduction (rounded to two decimal places):

\[
\begin{bmatrix}
1 & -.4 & -.6 & 0 \\
-.6 & .9 & -.2 & 0 \\
- .4 & -.5 & .8 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -.4 & -.6 & 0 \ \\
0 & .66 & -.56 & 0 \ \\
0 & -.66 & .56 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -.4 & -.6 & 0 \\
0 & .66 & -.56 & 0 \ \\
0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -.4 & -.6 & 0 \\
0 & 1 & -.85 & 0 \ \\
0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -.94 & 0 \\
0 & 1 & -.85 & 0 \ \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

General solution: \( p_C = .94 p_S, \ p_E = .85 p_S \) and \( p_S \) is free.

Equilibrium price vector:
\[ \mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94 p_S \\ .85 p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix} \]

Restriction: choose \( p_S \) non-negative, e.g.: \( p_S = \$100 \) million, \( p_C = \$94 \) million, \( p_E = \$85 \) million.

**Balancing Chemical Equations**

Propane gas burning:

\[ \text{C}_3\text{H}_8 \text{ (propane)} + \text{O}_2 \text{ (oxygen)} \rightarrow \]
\[ \text{CO}_2 \text{ (carbon dioxide)} + \text{H}_2\text{O} \text{ (water)} \]

Problem: how much propane and oxygen (exactly) produce how much carbon dioxide and water? Or:

\[ (x_1)\text{C}_3\text{H}_8 + (x_2)\text{O}_2 \rightarrow (x_3)\text{CO}_2 + (x_4)\text{H}_2\text{O} \]

Additional restriction: molecule amounts \( (x_i) \) must be integral

Solution: for each molecule (reactants and products), con-
The "atom vector" of atoms per molecule:
\[
\begin{bmatrix}
\text{Carbon} \\
\text{Hydrogen} \\
\text{Oxygen}
\end{bmatrix}
\]

We have:
\[
\begin{aligned}
\text{C}_3\text{H}_8 : & \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \\
\text{O}_2 : & \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \\
\text{CO}_2 : & \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \\
\text{H}_2\text{O} : & \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}
\end{aligned}
\]

Balancing, we require:
\[
\begin{aligned}
x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} &= x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}
\end{aligned}
\]

Rearranging:
\[
\begin{aligned}
x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\]

Row reduction . . . :
\[
\begin{aligned}
x_1 &= \frac{1}{4} x_4, \\
x_2 &= \frac{5}{4} x_4, \\
x_3 &= \frac{3}{4} x_4, \\
x_4 &\text{ free}
\end{aligned}
\]

Due to integral constraint, \( x_4 := 4 \) and we have:
\[
\text{C}_3\text{H}_8 + 5\text{O}_2 \rightarrow 3\text{CO}_2 + 4\text{H}_2\text{O}
\]
Note: since $x_4$ free, there exist an infinite number of solutions, i.e., every integral multiple of the above.

**Network Flow**

Studying flow through network:

1) urban planners and traffic engineers: traffic flow through city streets
2) electrical engineers: current flow through circuits
3) economists: distribution of products from manufacturers to consumers, through wholesalers and retailers

System can have 100s or 1000s of variables and equations

Network:

1) nodes (or junctions)
2) branches connecting some or all nodes

Flow:

1) amount or rate
2) along each branch
3) directed
4) known or variable \( (x_i) \)

Conservation assumption for network \textit{and} each node:

\[
\text{flow in} = \text{flow out}
\]

Conservation assumptions at nodes:
<table>
<thead>
<tr>
<th>Intersection</th>
<th>Flow in</th>
<th>=</th>
<th>Flow out</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$300 + 500$</td>
<td>=</td>
<td>$x_1 + x_2$</td>
</tr>
<tr>
<td>B</td>
<td>$x_2 + x_4$</td>
<td>=</td>
<td>$300 + x_3$</td>
</tr>
<tr>
<td>C</td>
<td>$100 + 400$</td>
<td>=</td>
<td>$x_4 + x_5$</td>
</tr>
<tr>
<td>D</td>
<td>$x_1 + x_5$</td>
<td>=</td>
<td>$600$</td>
</tr>
</tbody>
</table>

Conservation assumption for entire network:

$$500 + 300 + 100 + 400 = 300 + x_3 + 600$$

or: $x_3 = 400$

We have:

$$x_1 + x_2 = 800$$

$$x_2 - x_3 + x_4 = 300$$

$$x_4 + x_5 = 500$$

$$x_1 + x_5 = 600$$

Row reduction of augmented matrix:

$$x_1 + x_5 = 600$$

$$x_2 - x_5 = 200$$

$$x_3 = 400$$

$$x_4 + x_5 = 500$$
General pattern flow:

\[
\begin{align*}
  x_1 &= 600 - x_5 \\
  x_2 &= 200 + x_5 \\
  x_3 &= 400 \\
  x_4 &= 500 - x_5 \\
  x_5 &\text{ is free}
\end{align*}
\]

Note:

1) negative flow means in the opposite direction
2) limitation here: not possible, since only one way
3) leads to limitations, e.g.: \( x_5 \leq 500 \)
4) other limitations?