Marginal and Conditional Densities

Let a random point \((X, Y)\) be distributed in the triangular region \(\Delta\) bounded by 
\[x = 0, \ y = 0, \ \text{and} \ x + 2y = 2.\] 
Suppose that the density \(f = f_{X,Y}\) of \((X, Y)\) is 
\[f(x, y) = 3y, \quad (x, y) \in \Delta.\]
In other words, the chance of finding \((X, Y)\) in a tiny rectangle \([x, x + dx] \times [y, y + dy]\), 
positioned entirely within \(\Delta\), is \(f(x, y)dx\,dy = 3y\,dx\,dy\). The total probability mass is 
\[\iint_{\Delta} 3y\,dx\,dy = 1.\]
Note that, although the upper \(y\)-layers of \(\Delta\) are shorter than lower \(y\)-layers, they have a 
higher concentration of probability.

The marginal density of \(X\) is obtained by pushing all of the mass down onto the \(x\)-axis. 
\[f_X(x) = \int_y f(x, y)\,dy\]
\[= \int_0^{1-\frac{1}{2}x} 3y\,dy\]
\[= \frac{3}{2} \left(1 - \frac{1}{2} x\right)^2, \quad 0 < x < 2.\]
Here is the plot of the marginal density \(f_X(x)\):

Check that \(\int_0^2 f_X(x)\,dx = 1.\)
The marginal density of $Y$ is obtained by compressing the $y$-layers to the $y$-axis.

$$f_Y(y) = \int_x f(x, y)dx$$

$$= \int_0^{2-2y} 3y dx$$

$$= 6y(1-y), \quad 0 < y < 1.$$ 

Here is the plot of the marginal density $f_Y(y)$:

Check that $\int_0^1 f_Y(y)dy = 1$.

We may also calculate various conditional densities. For instance,

$$f_{Y|X}(y|1) = \frac{f(1, y)}{f_X(1)}$$

$$= \frac{3y}{3/8}$$

$$= 8y, \quad 0 < y < 1/2.$$ 

Check that $\int_0^{1/2} f_{Y|X}(y|1)dy = 1$. At the same time,

$$f_{X|Y}(x|1/2) = \frac{f(x, 1/2)}{f_Y(1/2)}$$

$$= \frac{3/2}{3/2}$$

$$= 1, \quad 0 < x < 1.$$