Multipliers of Gaussian elimination

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ be a matrix with many nonzero entries.

Let us row reduce $A$ to an echelon form. Since $A$ only has three rows, the process should take a maximum of $2+1 = 3$ steps. Along the way we will keep track of special parameters.

First column clean-up. Assume that $a_{11} \neq 0$ (otherwise, interchange rows if $a_{21}$ or $a_{31} \neq 0$).

$\quad$ Step 1 $R_2 + \left(-\frac{a_{21}}{a_{11}} \right) R_1 \quad \mu_{21} = -\frac{a_{21}}{a_{11}}$

$\quad$ Step 2 $R_3 + \left(-\frac{a_{31}}{a_{11}} \right) R_1 \quad \mu_{31} = -\frac{a_{31}}{a_{11}}$

$\quad$ Result $A^{(1)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} \end{bmatrix}$, where $a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$.

Second column clean-up. Assume that $a_{22}^{(1)} \neq 0$ (otherwise, interchange rows if $a_{32}^{(1)} \neq 0$).

$\quad$ Step 3 $R_3 + \left(-\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \right) R_2 \quad \mu_{32} = -\frac{a_{32}^{(1)}}{a_{22}^{(1)}}$

$\quad$ Result $A^{(2)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} \end{bmatrix}$, where $a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} a_{2j}$.

The reduction is complete, $A^{(2)}$ is an echelon form of $A$.

The numbers $\mu_{21}$, $\mu_{31}$, $\mu_{32}$ are the multipliers of the row reduction process.

The index of a multiplier indicates the position of the entry it is used to eliminate.

If the entry to eliminate is zero, the corresponding multiplier is assigned the zero value.

For an $m \times n$ matrix, we have $(m-1) + (m-2) + \ldots + 1 = \frac{1}{2} m(m-1)$ multipliers.

These multipliers carry all information on steps and may be arranged in a triangular table:

$\begin{array}{cccc}
\mu_{21} \\
\mu_{31} & \mu_{32} \\
\mu_{41} & \mu_{42} \\
\vdots & \vdots & \ddots & \mu_{m-1,m-2} \\
\mu_{m,1} & \mu_{m,2} & \mu_{m,m-2} & \mu_{m,m-1}
\end{array}$