## The Matrix of the Orthogonal Projection in the Plane

Let $\ell$ be a line through the origin, and let it make an angle of $\theta$ with the horizontal axis. Let us determine the matrix $A$ of the orthogonal projection $P$ onto $\ell$.

The columns of $A$ are given by the images of $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under $P$.

We can determine the projections of $e_1$ and $e_2$ onto $\ell$ geometrically.

Assuming that $\theta$ is acute, we have the following picture:

Observe that the length of $P(e_1)$ is $\cos \theta$, and so $P(e_1) = \begin{bmatrix} \cos^2 \theta \\ \cos \theta \sin \theta \end{bmatrix}$.

Similarly, the length of $P(e_2)$ is $\sin \theta$, and so $P(e_2) = \begin{bmatrix} \sin \theta \cos \theta \\ \sin^2 \theta \end{bmatrix}$.

It follows that the matrix of $P$ is

$$A = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}.$$ 

Note that $A$ is symmetric (its off-diagonal entries are equal), its diagonal entries are non-negative and sum to 1, and its reduced row-echelon form is $\begin{bmatrix} 1 & \tan \theta \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

**Example.** If $\ell$ passes through $(1,2)$, we have $\cos \theta = 1/\sqrt{5}$ and $\sin \theta = 2/\sqrt{5}$.

The matrix of $P$ in this case is $A = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$.

For instance, the projection of $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ onto $\ell$ is $P(x) = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$. 
