The most powerful test for the variance of a normal distribution

Let \( X_1, \ldots, X_n \) be a random sample from a normal distribution with known mean \( \mu \) and unknown variance \( \sigma^2 \). Suggested are two hypotheses: \( \sigma = \sigma_0 \) and \( \sigma = \sigma_1 \).

Let us derive the likelihood ratio criterion at significance level \( \alpha \), for each \( 0 < \alpha < 1 \).

Form the likelihood quotient,

\[
\mathcal{L}(x_1, \ldots, x_n) = \frac{P(X_1 \approx x_1, \ldots, X_n \approx x_n \mid \sigma = \sigma_0)}{P(X_1 \approx x_1, \ldots, X_n \approx x_n \mid \sigma = \sigma_1)}
\]

\[
= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_0}} e^{-\frac{(x_i - \mu)^2}{2\sigma_0^2}} dx_i \\
= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_1}} e^{-\frac{(x_i - \mu)^2}{2\sigma_1^2}} dx_i \\
= \left( \frac{\sigma_1}{\sigma_0} \right)^n \exp \left\{ \frac{\sigma_0^2 - \sigma_1^2}{2\sigma_0^2\sigma_1^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}.
\]

The condition \( \mathcal{L}(x_1, \ldots, x_n) < c \) is equivalent to \( \sum_{i=1}^{n} (x_i - \mu)^2 > \tilde{c} \), if \( \sigma_0 < \sigma_1 \), and to

\( \sum_{i=1}^{n} (x_i - \mu)^2 < \tilde{c} \), if \( \sigma_0 > \sigma_1 \), where \( \tilde{c} = \frac{2\sigma_0^2\sigma_1^2}{\sigma_0^2 - \sigma_1^2} \left( n \ln \frac{\sigma_0}{\sigma_1} + \ln c \right) \).

Thus \( \sum_{i=1}^{n} (X_i - \mu)^2 \) is the statistic of the test. The critical value \( \tilde{c} \) is determined by \( \alpha \).

Assume that \( \sigma_0 < \sigma_1 \).

Under the hypothesis \( \sigma = \sigma_0 \), we have \( \left( \frac{X_i - \mu}{\sigma_0} \right)^2 \sim \chi_1^2 \) and \( \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma_0} \right)^2 \sim \chi_n^2 \).

Therefore the condition

\[
\alpha = P(l(X_1, \ldots, X_n) < c \mid \sigma = \sigma_0) = P \left( \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma_0} \right)^2 > \frac{\tilde{c}}{\sigma_0^2} \mid \sigma = \sigma_0 \right)
\]

gives \( \tilde{c} = \sigma_0^2\chi_n^2(\alpha) \), where \( \chi_n^2(\alpha) \) is the \( \alpha \)-upper quantile for the chi-square distribution with \( n \) degrees of freedom. Note that \( \tilde{c} \) does not depend on \( \sigma_1 \).

Under the hypothesis \( \sigma = \sigma_1 \), we have \( \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma_1} \right)^2 \sim \chi_n^2 \), and therefore

\[
\beta = P(l(X_1, \ldots, X_n) \geq c \mid \sigma = \sigma_1) = P \left( \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma_1} \right)^2 \leq \frac{\tilde{c}}{\sigma_1^2} \mid \sigma = \sigma_1 \right).
\]

Hence \( \beta = F \left( \frac{\sigma_0^2}{\sigma_1^2} \chi_n^2(\alpha) \right) \), where \( F(x) \) is the cumulative distribution function of \( \chi_n^2 \).

Observe that \( \beta \leq 1 - \alpha \) is monotone decreasing in \( \alpha \) and \( \sigma_1 \), and monotone increasing in \( \sigma_0 \), and that \( \chi_n(1 - \beta) = \frac{\sigma_0}{\sigma_1} \chi_n(\alpha) \).
EXAMPLE Examined are size–25 random samples from a normal distribution with mean \( \mu = 0 \) and unknown variance \( \sigma^2 \). Consider the null hypothesis \( \sigma = 1 \) versus the alternative hypothesis \( \sigma = 2 \). What is the likelihood ratio (rejection) criterion at the significance level \( \alpha = 0.05 \)? Find the probability of a type II error.

Solution. If the null hypothesis \( \sigma = 1 \) is true, then \( \sum_{i=1}^{25} X_i^2 \sim \chi^2_{25} \).

Since \( 0.05 = P \left( \sum_{i=1}^{25} X_i^2 > \tilde{c} \left| \sigma = 1 \right. \right) \), we have \( \tilde{c} = \chi^2_{25}(0.05) = 37.65 \).

So the null hypothesis is to be rejected whenever \( \sum_{i=1}^{25} X_i^2 > 37.65 \).

The type II error probability is \( \beta = P \left( \sum_{i=1}^{25} X_i^2/4 \leq 37.65/4 \left| \sigma = 2 \right. \right) = P(\chi^2_{25} \leq 9.4125) = 0.002 \).