1. (10 points) You are curious about the annual salary of three of your close friends (I just name them A, B and C.) You somehow got the ‘insider information’ that the sum of the three quantities is 2 million, that A earns $10,000 more than B annually, and B earns $8000 more than C.

Can you determine the salaries of A, B and C? If so, determine them; otherwise explain why.

2. (20 points) Again, there are three quantities of interest which I call $x$, $y$ and $z$. And again we cannot directly acquire these three quantities, but this time you are given the indirect information that $x$ is a certain amount, say $a$ units, more than $y$, $x$ is $b$ units more than $z$, and $y$ is $c$ units more than $z$.

Under what condition(s) on $a$, $b$ and $c$ are there values of $x$, $y$ and $z$ that satisfy the above information? In this case, can you determine $x$, $y$ and $z$ uniquely?

Under what condition(s) on $a$, $b$ and $c$ would there be no values of $x$, $y$ and $z$ that satisfy the above information?

3. You have a certain fixed amount of fencing material, and you would like to use it to enclose the largest possible area. This is known as the isoperimetric problem.

This optimization problem is not easy; the difficulty hinges on the simple fact that there are too many possible shapes one can enclose with the same amount of fencing material: a triangle, a pentagon, a hexagon, an ellipse, or any shape with a curved boundary.

(5 points) The problem becomes solvable by freshman calculus (in fact you only need high-school algebra in this case) if we restrict ourselves to rectangular shapes. Among all rectangles with a fixed perimeter, which one has the largest area? Set up the optimization problem by defining appropriate variable(s) and show how to solve it analytically.

(25 points) Among all the triangles with a fixed perimeter, which triangle has the largest area? Hint: You can assume the three sides of a triangle have lengths $a$, $b$, $c$, with $a+b+c = 2$. So there there are two degrees of freedom. Use Heron’s formula.
What are the constraints on \((a, b)\)? Use the triangle inequality to determine the feasible region for \((a, b)\). Show that the constraints are given by the same kind of linear equalities as seen in a linear program. *Carefully graph the constraint set.*

(10 points) Is the optimum value of the problem attained at a *vertex* of the feasible region, as you may expect from a linear program?

4. Use the following Matlab code to generate \((a_i, b_i), i = 1, \ldots, 11:\)

```matlab
slope=3; intercept=-2;
abscissa = (-5:5)'; m = length(abscissa);
WhiteNoise = 5*randn(m,1);
ordinates = slope*abscissa + intercept + WhiteNoise;

GrossError=80;
ordinates(6)=ordinates(6)+GrossError;
ordinates(10)=ordinates(10)-GrossError;
```

The goal is to recover the slope \(x\) and intercept \(\gamma\) from the noisy data.

As we discussed in class, two ways to formulate this problem are to solve:

- **[Least \(L^2\) error]** \[\min_{x,\gamma} \sum_{i=1}^{m} (xa_i + \gamma - b_i)^2\]
- **[Least \(L^1\) error]** \[\min_{x,\gamma} \sum_{i=1}^{m} |xa_i + \gamma - b_i|\]

Both approaches have an elegant solution. As discussed in class and in Section 1.3.2 of the textbook, the \(L^1\) approach can be formulated as a linear program.

(30 points) For the least square problem, show that the optimization problem can be solved based on solving a \(2 \times 2\) system of linear of linear equations. Code up the problem and solve it using the linear solver \(\backslash\) in Matlab, and apply it to the data above.

For the least \(L^1\) error problem, show that the optimization problem is a linear program. Code up the problem and solve it using the LP solver \texttt{linprog()} in the optimization toolbox. Apply it to the same data.

Create a plot that compares the two solutions.

For full credits, turn in both the code and the graphical outputs.