Math 121: Calculus 1 - Fall 2012/2013
Review of Precalculus Concepts

Introduction

Welcome to Math 121 - Calculus 1, Fall 2012/2013! This problems in this packet are designed to help you review the topics from Algebra and Precalculus which are important to your success in Calculus. When you come across a problem which requires a little review, please revisit an appropriate section of your textbook or ask your instructor for some advice/references. Much of this material is covered in Chapter 0 and Appendix B of the textbook.

Any questions regarding your readiness for Math 121 should be resolved immediately. Please speak with your instructor and your academic advisor should you have any concerns about your placement in this course. The deadline to add/drop a course is Friday, 9/28 at 5 pm.

Preliminaries

QUICK REVIEW: Incoming Calculus I students should be familiar with the following facts about the rectangular coordinate system.

- Let $A$, $B$, and $C$ be the lengths of the three sides of a right triangle where $C$ is the length of the hypotenuse. Then, the Pythagorean Theorem says that $A^2 + B^2 = C^2$.
- The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A circle with a center of $(h, k)$ and a radius of $r$ has an equation of $(x-h)^2 + (y-k)^2 = r^2$.
- You should be familiar with basic area, volume, and perimeter formulas from geometry.

PRACTICE PROBLEMS:

1. Use the Pythagorean Theorem and the distance formula to show that the points $(4, 0)$, $(2, 1)$, and $(-1, -5)$ are vertices of a right triangle.

2. For each of the following, determine an equation of the circle which has the given characteristics:
   
   (a) Center: $(2, -1)$; Radius: 4
   (b) Endpoints of a diameter: $(-3, -2)$ and $(-7, 8)$

3. A "Slow Moving Vehicle" sign has the shape of an equilateral triangle. The sign has a perimeter of 120 centimeters.
(a) Find the length of each side of the sign.
(b) Find the area of the sign.

4. A rectangular room is 1.5 times as long as it is wide. Its perimeter is 25 meters. Find the dimensions of the room.

Lines

QUICK REVIEW: Incoming Calculus I students should be familiar with the following facts about lines.

• The slope of a line through the points \((x_1, y_1)\) and \((x_2, y_2)\) is given by \(m = \frac{y_2 - y_1}{x_2 - x_1}\).

• The slope of any horizontal line is 0 and the slope of any vertical line is undefined.

• The Point-Slope Form of a line with a slope of \(m\) which passes through the point \((x_1, y_1)\) is \(y - y_1 = m(x - x_1)\).

• The Slope-Intercept Form of a line with a slope of \(m\) and a \(y\)-intercept of \(b\) is \(y = mx + b\). Note: a \(y\)-intercept of \(b\) means that the graph passes through the point \((0, b)\).

• The equation of the horizontal line which passes through the point \((a, b)\) is \(y = b\) and the equation of the vertical line which passes through the point \((a, b)\) is \(x = a\).

• Two lines are parallel is they have the same slope OR they are both vertical.

• Two lines are perpendicular is the product of their slopes is \(-1\) (i.e., their slopes are negative-reciprocals) OR one is vertical whereas the other is horizontal.

WORKED EXAMPLES

1. Find an equation of the line which passes through the point \((1, -2)\) and which has a slope of 5.

Solution #1: We are given enough information to use slope form. Specifically, we know that \((x_1, y_1) = (1, -2)\) and \(m = 5\). Thus \(y - (-2) = 5(x - 1)\). If we simplify this, we get \(y = 5x - 7\).

Solution #2: We could have used slope-intercept form instead. Specifically, since \(m = 5\), we have that \(y = 5x + b\). The equation must hold for all points on the line; so, if we substitute in \((x, y) = (1, -2)\), we can solve for \(b\). I.e., \(-2 = 5(1) + b\) which gives us that \(b = -7\). Thus, the equation of the specified line is \(y = 5x - 7\), as before.
2. Find an equation of the line which passes through (1, 3) and is perpendicular to the line 3x + 6y = 5.

Solution: First, we find the slope of the given line 3x + 6y = 5. One way to do this is to re-write the equation in slope-intercept form: \( y = -\frac{1}{2}x + \frac{5}{2} \). We now see that the given line has a slope of \(-\frac{1}{2}\) and realize that any line perpendicular to it must have a slope of 2. Then, the equation of the requested line in point-slope form is \( y - 3 = 2(x - 1) \), which is equivalent to the slope-intercept form of \( y = 2x + 1 \).

PRACTICE PROBLEMS:

1. For each of the following, write the equation of the line in slope-intercept form (where appropriate) which satisfies the given condition.

   (a) The line which passes through (1, 2) and (5, 7).
   (b) The line which has an x intercept of 5 and a y intercept of 3.
   (c) The line which is passes through (−5, 9) and (4, 9).
   (d) The line which is parallel to \( y = 4x - 7 \) and passes through \( \left( \frac{1}{4}, 2 \right) \).
   (e) The line which is perpendicular to \( y = 4x - 7 \) and passes through \( \left( \frac{1}{4}, 2 \right) \).
   (f) The line which is perpendicular to \( y = 3 \) and passes through the (1, 0)

Functions

QUICK REVIEW: Incoming Calculus I students should be familiar with the following facts about functions.

- A function from a set A to a set B is a relation (rule) which assigns to each element \( x \) in A exactly one element \( y \) in the set B. Notation: \( y = f(x) \)
- The domain of a function is the set of all allowable inputs (set A in the above definition). The range of a function is the set of all resulting output values (set B in the above definition).
- The Vertical Line Test says that a given graph represents a function is any vertical line intersects the graph at most once. So, if you can draw a vertical line which intersects a given graph two or more times, the graph does not represent a function.
- Let \( f \) and \( g \) be functions. We define:
  - \((f + g)(x) = f(x) + g(x)\)
  - \((f - g)(x) = f(x) - g(x)\)
  - \((fg)(x) = f(x)g(x)\)
• \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \)

• The **composition** of the function \( f \) with \( g \) is denoted \( f \circ g \). This function is defined as \( (f \circ g)(x) = f(g(x)) \). In words, the output of the first function (\( g \) in this case) is the input of a second function (\( f \) in this case). Notice that we can illustrate composition \( f \circ g \) as follows:

\[
x \rightarrow g \rightarrow g(x) \rightarrow f \rightarrow f(g(x))
\]

**CAUTION:** In general, \( f \circ g \neq g \circ f \). In words, the order of the composition is usually important.

• Functions \( f \) and \( g \) are inverse functions if \( f(g(x)) = x \) for all \( x \) in the domain of \( g \)

AND \( g(f(x)) = x \) for all \( x \) in the domain of \( f \). Notation: the inverse function of \( f \) is written \( f^{-1}(x) \).

• The domain of \( f \) is the range of \( f^{-1} \) and the range of \( f \) is the domain of \( f^{-1} \).

• the **Horizontal Line Test** says that a function \( f \) has an inverse function \( f^{-1} \) if and only if any horizontal line intersects the graph of \( f \) at most once. If you can draw a horizontal line which intersects the graph of \( f \) more than once, then \( f^{-1} \) does not exist on the given domain.

**EXPECTED SKILLS:** As a calculus student, you are expected to...

• Know the definition of a function.

• Be able to determine if a curve in the \( xy \)-plane represents the graph of a function \( y = f(x) \).

• Know how to evaluate a function at specified values.

• Be able to compute the domain and range of a function.

• Have the skills needed to write formulas for functions in the context of word problems.

• Be able to combine functions algebraically or via a composition of functions.

• Be able to determine whether a function has an inverse, and, if so, determine the inverse function.

• Know how to the domain and range of a function and its inverse are related as well as the relationship between the graph of a function and its inverse.
PRACTICE PROBLEMS:

1. Which of the following graphs represents $y$ as a function of $x$?

(a) (c) (b) (d)

For numbers 2-3, evaluate the function at the given input.

2. $f(x) = 2x - 3$

(a) $f(1)$
(b) $f(-3)$
(c) $f(x - 1)$

3. $f(x) = \begin{cases} 
3x - 1 & \text{if } x < -1 \\
4 & \text{if } -1 \leq x \leq 1 \\
x^2 & \text{if } x > 1 
\end{cases}$

(a) $f(-2)$
(b) $f \left( -\frac{1}{2} \right)$
(c) $f(3)$
(d) $f(3t)$
4. Let \( f(x) = x^2 - 2x \). Compute and simplify the difference quotient \( \frac{f(x + h) - f(x)}{h} \).

5. Let \( f(x) = \frac{1}{x^2} \). Compute and simplify \( \frac{f(x) - f(3)}{x - 3} \).

For problems 6-11, compute the domain of the given function. Express your answer as an inequality and in interval notation.

6. \( f(x) = 5x^2 - 3x + 1 \)

7. \( \frac{3x}{x - 5} \)

8. \( \sqrt{10 - x} \)

9. \( \sqrt{10 - x^2} \)

10. \( \sqrt[3]{10 - x} \)

11. \( \frac{1}{x} - \frac{1}{x - 2} \)

12. Write the area \( A \) of a square as a function of its perimeter \( P \).

For problems 13-14, compute \( f \circ g \), \( g \circ f \), and \( f \circ f \)

13. \( f(x) = x^2 \), \( g(x) = x - 1 \)

14. \( f(x) = \sqrt{x + 1} \), \( g(x) = x^2 + 5 \)

15. Let \( f(x) \) and \( g(x) \) defined as in the table below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Compute each of the following:

(a) \( (f \circ g)(4) \)
(b) \( g(f(4)) \)
(c) \( f(g(1)) \)
(d) \( (g \circ g)(4) \)

For problems 16-18, compute \( f^{-1} \). Also state the domain and range of \( f^{-1} \) in interval notation.

16. \( f(x) = 3x + 1 \)
17. \( f(x) = \sqrt[3]{x - 1} \)

18. \( f(x) = x^2 + 9, x \geq 0 \)

19. Determine whether the following functions have inverses.

(a) \[ \text{Graph of a linear function.} \]

(b) \[ \text{Graph of a quadratic function.} \]

(c) \[ \text{Graph of a cubic function.} \]

(d) \[ \text{Graph of a square root function.} \]

Exponents & Logs

EXPECTED SKILLS: As a calculus student, you are expected to...

- Know the definition of exponential and logarithmic functions as well as their graphs.
- Know the algebraic properties of exponents and logarithms.
- Be able to solve exponential and logarithmic equations.

PRACTICE PROBLEMS

1. Simplify:

   (a) \( \left( \frac{49}{100} \right)^{-3/2} \)

   (b) \( (5a^{2/3})(4a^{3/2}) \)

   (c) \( (4a^{5/3})^{3/2} \)

   (d) \( e^{\ln^3} \)
2. For each of the following, use properties of logarithms to expand (as much as possible) the given expression as a sum, difference, and or constant multiple of logarithms. (Assume that all variables are positive)

(a) \( \log_5 (5x^2 \sqrt{y}) \)
(b) \( \ln \sqrt{x^3(x^2 + 3)} \)

3. Solve for \( x \). Where appropriate, you may leave your answer in logarithmic form.

(a) \( e^x + 5 = 60 \)
(b) \( 3^{x-5} - 4 = 11 \)
(c) \( 2 \log_5 (3x) = 4 \)
(d) \( \log_3 x + \log_3 (x - 8) = 2 \)
(e) \( \frac{1 + \ln x}{2} = 0 \)

4. The equation \( Q(t) = 30e^{-4t} \) gives the mass (in grams) of a radioactive element that will remain from some initial quantity after \( t \) hours of radioactive decay.

(a) How many grams were present initially?
(b) How long will it take for 40\% of the element to decay? (You may leave your answer in logarithmic form.)

**Trigonometry**

EXPECTED SKILLS: As a calculus student, you are expected to...

- Understand how to convert between the degree and radian measurements of an angle.
- Know the basic trigonometric identities found on the “Course Documents” page of the main course website.
- Know the domain and range for sine and cosine. And, be able to use this information to find the domain and range of tangent, secant, cosecant, cotangent.
• Be able to evaluate all trigonometric functions as the "Special Angles." In other words, you should be able to evaluate the trigonometric functions at angles with radian measure of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$, and all related angles in different quadrants. Common methods for doing this include using the unit circle or special triangles.

• Know the graphs of sine, cosine, tangent, secant, cosecant, and cotangent. Also, know the periods for these functions.

• Be able to solve trigonometric equations.

• Understand the concept of inverse trigonometric functions (especially $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$). In particular, you should know the range of these functions.

• Know the graph of the inverse tangent function.

PRACTICE PROBLEMS

1. Convert the following angles from degrees to radians.
   (a) 115°
   (b) −150°

2. Convert the following angles from radians to degrees.
   (a) $\frac{\pi}{9}$
   (b) $\frac{2\pi}{3}$

3. Determine the quadrant in which the terminal side of each angle lies. (Each angle is measured in radians)
   (a) $\frac{\pi}{5}$
   (b) $\frac{11\pi}{8}$

4. Evaluate (if possible) all six trigonometric functions at the given angle.
   (a) 0°
   (b) 30°
   (c) $\frac{\pi}{4}$
   (d) $\frac{\pi}{3}$
(e) $\frac{\pi}{2}$
(f) $\frac{5\pi}{6}$
(g) $-\frac{5\pi}{6}$
(h) $225^\circ$
(i) $300^\circ$
(j) $-90^\circ$

5. Solve for $x$:

6. Solve the following by drawing a triangle.

   (a) Find all possible values of $\sin \theta$ and $\cos \theta$ given that $\tan \theta = 3$.
   
   (b) Find all possible values of $\tan \theta$ and $\csc \theta$ given that $\sec \theta = \frac{5}{2}$.

7. Compute $\cos \theta$ if $\sin \theta = -\frac{3}{5}$ and the terminal side of $\theta$ is in Quadrant IV.

8. Use the given information to find the exact values of the remaining five trigonometric functions of $\theta$.
(a) \( \cos \theta = \frac{3}{5} \) and \( 0 < \theta < \frac{\pi}{2} \)

(b) \( \tan \theta = -\frac{1}{3} \) and \( -\frac{\pi}{2} < \theta < 0 \)

9. Find two values of \( \theta \) in \([0, 2\pi)\) which satisfy the given equation:

(a) \( \sin \theta = \frac{1}{2} \)

(b) \( \cos \theta = \frac{\sqrt{2}}{2} \)

(c) \( \tan \theta = 1 \)

10. Evaluate each of the following. (Be careful with the ranges of the inverse trigonometric functions!)

(a) \( \arcsin \frac{1}{2} \)

(b) \( \arccos \frac{1}{2} \)

(c) \( \arctan \frac{\sqrt{3}}{3} \)

(d) \( \cos^{-1} \left( -\frac{\sqrt{3}}{2} \right) \)

(e) \( \arctan (-\sqrt{3}) \)

(f) \( \sin^{-1} \left( -\frac{\sqrt{3}}{2} \right) \)

(g) \( \tan^{-1} 1 \)

(h) \( \tan^{-1} (-1) \)

11. Use an inverse trigonometric function to express \( \theta \) as a function of \( x \):

(a) \[ \theta = \arctan \left( \frac{x}{4} \right) \]

(b) \[ \theta = \arcsin \left( \frac{2x}{x+3} \right) \]
12. Find the exact value of each expression. (HINT: sketch a right triangle)

(a) $\sin \left( \tan^{-1} \frac{3}{4} \right)$
(b) $\sec \left( \arctan \left( -\frac{3}{5} \right) \right)$
(c) $\sin \left( \arccos \left( -\frac{2}{3} \right) \right)$
(d) $\csc \left( \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \right)$

13. Find the exact value of each expression.

(a) $\sin^{-1} \left( \sin \left( \frac{\pi}{3} \right) \right)$
(b) $\sin^{-1} \left( \sin \left( \frac{2\pi}{3} \right) \right)$
(c) $\cos^{-1} \left( \cos \left( \frac{\pi}{4} \right) \right)$
(d) $\cos^{-1} \left( \cos \left( -\frac{\pi}{4} \right) \right)$
(e) $\tan^{-1} \left( \tan \left( \frac{\pi}{6} \right) \right)$
(f) $\tan^{-1} \left( \tan \left( \frac{5\pi}{6} \right) \right)$

14. Solve the following equations.

(a) $\sin x + \sqrt{2} = -\sin x$
(b) $3 \tan^2 x - 1 = 0$
(c) $\cot x \cos^2 x = 2 \cot x$
(d) $2 \sin^2 x + 3 \sin x + 1 = 0$
(e) $2 \sin^2 x - \cos x = 1$
(f) $2 \cos (3x) - 1 = 0$

15. For each of the following, find all solutions in the interval $[0, 2\pi)$

(a) $\sqrt{3} \csc x - 2 = 0$
(b) $3 \cot^2 x - 1 = 0$
(c) \((3\tan^2 x - 1)(\tan^2 x - 3) = 0\)
(d) \(\cos^3 x = \cos x\)
(e) \(\cos x + \sin x \tan x = 2\)
(f) \(3\sec^2 x - 4 = 0\)
(g) \(\cos (2x)(2\cos x + 1) = 0\)

**Graphs of Elementary Functions**

**PRACTICE PROBLEMS:**

1. For each of the following, (i) Sketch the given function, labeling all intersections with the coordinate axes; (ii) Label all horizontal/vertical asymptotes, if any exist; (iii) Express the domain and range in interval notation.

   (a) The Identity Function, \(y = x\)
   (b) The Square Function, \(y = x^2\)
   (c) The Cube Function, \(y = x^3\)
   (d) The Absolute Value function, \(y = |x|\).  (Also, express this function as a "Piecewise Function".)
   (e) The Square-Root Function, \(y = \sqrt{x}\)
   (f) The Cube-Root Function, \(y = \sqrt[3]{x}\)
   (g) The Natural Log Function, \(y = \ln x\)
   (h) The Exponential Function, \(y = e^x\)
   (i) The Inverse Tangent Function, \(y = \tan^{-1} x = \arctan x\)

2. For each of the following, (i) Sketch the given function on the interval \([-2\pi, 2\pi]\). Label all intersections with the coordinate axes; (ii) Label all horizontal/vertical asymptotes, if any exist; (iii) Express the domain and range in interval notation.

   (a) The Sine Function, \(y = \sin x\)
   (b) The Cosine Function, \(y = \cos x\)
   (c) The Tangent Function, \(y = \tan x\)
   (d) The Cotangent Function, \(y = \cot x\)
   (e) The Secant Function, \(y = \sec x\)
   (f) The Cosecant Function, \(y = \csc x\)