Math 200 - Final Exam - June 7th, 2016

Name: Solution

The following rules apply:

- **This is a closed-book exam.** You may *not* use any books or notes on this exam.

- **For free response questions, you must show all work.** Answers without proper justification will not receive full credit. Partial credit will be awarded for significant progress towards the correct answer. Cross off any work that you do not want graded.

- **For multiple choice questions, circle the letter of the best answer.** Make sure your circles include just one letter. These problems will be marked as correct or incorrect; partial credit will not be awarded for problems in this section.

- **You have 50 minutes to complete this exam.** When time is called, stop writing immediately and turn in your exam to the nearest proctor.

- **You may not use any electronic devices including (but not limited to) calculators, cell phone, or iPods.** Using such a device will be considered a violation of the university's academic integrity policy and, at the very least, will result in a grade of 0 for this exam.

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Part I: Free Response

1. (12 points) Let $G$ be the wedge in the first octant that is cut from $y^2 + z^2 \leq 1$ by $y = x$ and $x = 0$. Find the volume of the solid $G$ using a triple integral.

\[
\text{Volume of } G = \iiint_G 1 \, dv
\]

\[
= \iiint_R \sqrt{1-y^2} \, dA
\]

\[
= \int_0^1 \int_y^1 \int_0^{\sqrt{1-y^2}} 1 \, dz \, dx \, dy
\]

\[
= \int_0^1 \int_y^1 \sqrt{1-y^2} \, dy \, dx
\]

Switch to Type II:

\[
= \int_0^1 \int_0^y \sqrt{1-y^2} \, dx \, dy
\]

\[
= \int_0^1 \left[ x \sqrt{1-y^2} \right]_0^y \, dy
\]

\[
= \int_0^1 \left[ \frac{1}{2} \cdot y \cdot \sqrt{1-y^2} \right] \, dy
\]

\[
= -\frac{1}{2} \left[ \frac{2}{3} \cdot y^\frac{3}{2} \right]_0^1
\]

\[
= -\frac{1}{3} \left[ 0 - 1 \right] = \frac{1}{3}
\]
2. Consider the surfaces \( S_1 : z = x^2 - y^2 \) and \( S_2 : y^2 + z^2 = 10 \)

(a) (10 points) Find an equation of the tangent line to the curve of intersection of \( S_1 \) and \( S_2 \) at the point \((2, 1, 3)\).

(b) (5 points) Find the acute angle between the planes which are tangent to the surfaces \( S_1 \) and \( S_2 \) at the point \((2, 1, 3)\).

\[ \nabla f(x, y, z) = (-2x, 2y, 1) \quad \nabla g(x, y, z) = (0, 2y, 2z) \]

\[ \nabla f(2, 1, 3) = (-4, 2, 1) \quad \nabla g(2, 1, 3) = (0, 2, 6) \]

\[ \mathbf{N} = \nabla f(2, 1, 3) \times \nabla g(2, 1, 3) \]

\[ = \begin{vmatrix} i & j & k \\ -4 & 2 & 1 \\ 0 & 2 & 6 \end{vmatrix} = (12 - 2)i - (-24 - 0)j + (-8 - 0)k \]

\[ = 10i + 24j - 8k \]

So the tangent line \( \mathbf{l}(t) \) is

\[ \mathbf{l}(t) = \langle 2, 1, 3 \rangle + t \mathbf{N} \]

\[ = \langle 2, 1, 3 \rangle + t \langle 10, 24, -8 \rangle \]

or

\[ x(t) = 2 + 10t, \quad y(t) = 1 + 24t, \quad z(t) = 3 - 8t \]

\( \cos \theta = \frac{\langle -4, 2, 1 \rangle \cdot \langle 0, 2, 6 \rangle}{\| \langle -4, 2, 1 \rangle \| \| \langle 0, 2, 6 \rangle \|} \]

\[ = \frac{-4 \cdot 0 + 2 \cdot 2 + 1 \cdot 6}{\sqrt{16 + 4 + 1}} = \frac{10}{\sqrt{21}} \]

\[ \Rightarrow \theta = \cos^{-1} \left( \frac{1}{\sqrt{21}} \right) \]
3. (15 points) Evaluate \( \iint_R (x+y)^2 \, dA \) where \( R \) is the region bounded by \( x + y = 0, x + y = 1, 2x - y = 0, \) and \( 2x - y = 3 \) using the following change of variables

\[
\begin{align*}
\begin{cases}
  u = x + y \\
  v = 2x - y
\end{cases} &\iff
\begin{cases}
  x = \frac{1}{3}u + \frac{1}{3}v \\
  y = \frac{2}{3}v - \frac{1}{3}v
\end{cases} \\
0 \leq u \leq 1 \\
0 \leq v \leq 3
\end{align*}
\]

\[
\begin{vmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{2}{3} & -\frac{1}{3}
\end{vmatrix} = \frac{1}{9} - \frac{2}{9} = -\frac{1}{3}
\]

Then

\[
\iint_R (x+y)^2 \, dA = \iint_{R'} u^2 \left| J(u,v) \right| \, dA' 
\]

\[
= \int_0^1 \int_0^3 u^2 \left| -\frac{1}{3} \right| \, dv \, du 
\]

\[
= \int_0^1 \int_0^3 u^2 \cdot \frac{1}{3} \cdot 3 \, du 
\]

\[
= \int_0^1 u^2 \, du = \frac{1}{3} \left[ u^3 \right]_0^1 = \frac{1}{3}
\]
4. Consider the solid $G$ enclosed by the sphere $x^2 + y^2 + z^2 = 25$.

(a) (5 points) Set up **but do not evaluate** $\iiint_G (x + y)\,dV$ using rectangular coordinates with the order of integration $dy\,dz\,dx$.

$$\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \int_{-\sqrt{25-x^2-z^2}}^{\sqrt{25-x^2-z^2}} (x+y)\,dy\,dz\,dx$$

(b) (5 points) Set up **but do not evaluate** $\iiint_G (x + y)\,dV$ using cylindrical coordinates.

$$\int_{0}^{2\pi} \int_{0}^{\sqrt{25-r^2}} \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} \left(r\cos\theta + r\sin\theta\right) r\,dz\,dr\,d\theta$$

(c) (5 points) Set up **but do not evaluate** $\iiint_G (x + y)\,dV$ using spherical coordinates.

$$\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{5} \left(\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta\right) \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
5. (15 points) Consider the function \( f(x, y) = x^4 - y^4 - 8x^2 + 4y \). Find all critical points for \( f \) and classify each as a relative minimum, relative maximum, or saddle point.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4x^3 - 16x = 0 \\
\frac{\partial f}{\partial y} &= -4y^3 + 4 = 0 \\
\end{align*}
\]

\( \Rightarrow (0, 1), (2, 1), (-2, 1) \): critical points.

\[
D(x, y) = \left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = (12x^2 - 16)(-12y^2)
\]

\[
\begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} = 12x^2 - 16 & \frac{\partial^2 f}{\partial x \partial y} = 0 \\
\frac{\partial^2 f}{\partial y^2} = -12y^2
\end{pmatrix}
= -144x^2y^2 + 192y^2.
\]

(i) \((0, 1)\)

\( D(0, 1) = 192 > 0 \), \( \frac{\partial^2 f}{\partial x^2}(0, 1) = -16 < 0 \)

relative maximum

(ii) \((2, 1)\)

\( D(2, 1) < 0 \); Saddle point

(iii) \((-2, 1)\)

\( D(-2, 1) < 0 \); Saddle point
Part II: Multiple Choice

6. (4 points) A portion of the surface defined by $z = f(x, y)$ is shown below.

Use the tangent lines in the figure to determine $\frac{\partial z}{\partial x}(1, 2)$.

(a) 2
(b) 1
(c) $\frac{1}{2}$
(d) $-\frac{1}{2}$
(e) $-1$

7. (4 points) Which of the following is an equation of the tangent line to $\vec{r}(t) = (e^t, e^{-t}, 1)$ at the point $(e, e^{-1}, 1)$?

(a) $\vec{T}(t) = (e, e^{-1}, 1) + t(e, e^{-1}, 0)$
(b) $\vec{T}(t) = (e, e^{-1}, 1) + t(e, -e^{-1}, 1)$
(c) $\vec{T}(t) = (e, e^{-1}, 1) + t(e^t, -e^{-t}, 0)$
(d) $\vec{T}(t) = (e, e^{-1}, 1) + t(e, -e^{-1}, 0)$
(e) $\vec{T}(t) = (e, e^{-1}, 1) + t(1, 1, 0)$
8. (4 points) Which of the following numbers is the area of the triangle with vertices \( P(1, 0, 0), \ Q(0, 2, 0), \) and \( R(0, 0, 3). \)
   (a) \( \frac{7}{2} \)
   (b) \( \frac{1}{2} \)
   (c) \( \frac{1}{3} \)
   (d) \( \frac{2}{3} \)
   (e) \( \frac{7}{2} \)

9. (4 points) Let \( f(x, y) = \ln \sqrt{x^2 + y^2}. \) At the point \( (3, 4, \ln 5) \) the maximum value of the directional derivative \( D_\alpha f(x, y) \) is
   (a) 1
   (b) 2
   (c) \( \sqrt{1 + \ln 25} \)
   (d) \( \sqrt{2} \)
   (e) \( \frac{1}{5} \)

10. (4 points) Which of the following iterated integrals is equivalent to \( \int_0^\pi \int_0^3 r^2 \, dr \, d\theta \)?
    (a) \( \int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2) \, dy \, dx \)
    (b) \( \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} \, dy \, dx \)
    (c) \( \int_0^3 \int_{\sqrt{9-x^2}}^{-\sqrt{9-x^2}} \sqrt{x^2 + y^2} \, dy \, dx \)
    (d) \( \int_{-3}^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2) \, dy \, dx \)
    (e) \( \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} \, dy \, dx \)
11. (4 points) Suppose \( \vec{v} \) and \( \vec{w} \) are arbitrary nonzero vectors, and \( k \) is an arbitrary scalar. Which of the following is false?

(a) \( \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \)

(b) \( \vec{v} \times \vec{w} = -(\vec{w} \times \vec{v}) \)

(c) \( \vec{v} \cdot (\vec{w} \times \vec{v}) = 0 \)

(d) \( \vec{w} \times (k\vec{w}) = 0 \)

(e) \( ||k\vec{w}|| = k||\vec{w}|| \)

12. (4 points) Which of the following results from switching the order of integration for the following iterated integral?

\[ \int_{-3}^{3} \int_{0}^{3} f(x, y) \, dy \, dx + \int_{0}^{3} \int_{x}^{3} f(x, y) \, dy \, dx \]

(a) \( \int_{0}^{3} \int_{-y}^{y} f(x, y) \, dx \, dy \)

(b) \( \int_{0}^{3} \int_{-3}^{y} f(x, y) \, dx \, dy \)

(c) \( \int_{-3}^{3} \int_{-y}^{y} f(x, y) \, dx \, dy \)

(d) \( \int_{-3}^{3} \int_{-3}^{y} f(x, y) \, dx \, dy \)

(e) \( \int_{0}^{3} \int_{-3}^{0} f(x, y) \, dy \, dx + \int_{x}^{3} \int_{0}^{3} f(x, y) \, dy \, dx \)