Generators of quantum Markov Semigroups

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Virginia Operator Theory and Complex Analysis Meeting (VOTCAM)
November 7th, 2015
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The $\sigma$-weak topology

Let $\mathcal{A}$ be a von Neumann algebra. Then $\mathcal{A}$ has a predual $\mathcal{A}^*$ and the $\sigma$-weak topology on $\mathcal{A}$ is the $\sigma(\mathcal{A}, \mathcal{A}^*)$ topology, that is, the weak* topology when $\mathcal{A}$ is viewed as the dual of $\mathcal{A}^*$.

Note 1: Every von Neumann algebra (when viewed as a Banach space) has a predual (Sakai).

Note 2: We are mostly interested in the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space. In this case $\mathcal{B}(\mathcal{H})^* = S_1(\mathcal{H})$. 
Completely positive operators

Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a bounded linear operator. Let $\mathcal{B}(\mathcal{H}) \otimes M_n$ be the $*$-algebra of $n \times n$ matrices with coefficients in $\mathcal{B}(\mathcal{H})$. We say the operator $T$ is **completely positive** if for any $n \in \mathbb{N}$, any positive element $[A_{ij}]_{1 \leq i, j \leq n} \in \mathcal{B}(\mathcal{H}) \otimes M_n$, and any $h_1, h_2, \ldots, h_n \in \mathcal{H}$ we have

$$\sum_{i,j=1}^{n} \langle h_i, T(A_{ij})h_j \rangle \geq 0.$$
Quantum dynamical semigroup

Let $\mathcal{A}$ be a von Neumann algebra. A **quantum dynamical semigroup (QDS)** is a one-parameter family $(T_t)_{t \geq 0}$ of $\sigma$-weakly continuous, completely positive, linear operators on $\mathcal{A}$ such that

(i) $T_0 = 1$

(ii) $T_{t+s} = T_t T_s$

(iii) for a fixed $A \in \mathcal{A}$, the map $t \mapsto T_t(A)$ is $\sigma$-weakly continuous.

Further, if $T_t(1) = 1$ for all $t \geq 0$ then we say the quantum dynamical semigroup is **Markovian** or we simply refer to it as a **quantum Markov semigroup (QMS)**. If the map $t \mapsto T_t$ is norm continuous then we say the semigroup is **uniformly continuous**.
Generator of a QMS

Given a QDS \((T_t)_{t \geq 0}\), we say that an element \(A \in \mathfrak{A}\) belongs to the domain of the infinitesimal generator \(L\) of \((T_t)_{t \geq 0}\), denoted by \(D(L)\), if

\[
\lim_{t \to 0} \frac{1}{t} (T_tA - A)
\]

converges in the \(\sigma\)-weak topology and, in this case, define the **infinitesimal generator** to be the generally unbounded operator \(L\) such that

\[
L(A) = \sigma\text{-weak-} \lim_{t \to 0} \frac{1}{t} (T_tA - A) , \quad A \in D(L).
\]

If \((T_t)_{t \geq 0}\) is uniformly continuous then the generator \(L\) is bounded and given by

\[
L = \lim_{t \to 0} \frac{1}{t} (T_t - 1)
\]

where the limit is taken in the norm topology. In this case \(T_t = e^{tL}\).
Lindblad (’76)

If \((T_t)_{t \geq 0}\) is a uniformly continuous QMS on \(\mathcal{B}(\mathcal{H})\) then there exists \(G \in \mathcal{B}(\mathcal{H})\) and a completely positive map \(\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\) such that the infinitesimal generator \(L\) of \((T_t)_{t \geq 0}\) is given by

\[
L(A) = \phi(A) + GA + AG^*
\]

for all \(A \in \mathcal{B}(\mathcal{H})\).

Note 1: Lindblad proved this for a uniformly continuous QMS on a hyperfinite factor \(\mathcal{A}\) of \(\mathcal{B}(\mathcal{H})\) (which includes the case \(\mathcal{A} = \mathcal{B}(\mathcal{H})\) by Topping (’71)).

Note 2: Christensen and Evans proved this for uniformly continuous QMS on arbitrary von Neumann algebras in ‘79.
Stinespring (‘55)

Let $\mathcal{B}$ be a $C^*$-subalgebra of the algebra of all bounded operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{A}$ be a $C^*$-algebra with unit. A linear map $T : \mathcal{A} \rightarrow \mathcal{B}$ is completely positive if and only if it has the form

$$T(A) = V^* \pi(A) V$$

where $(\pi, \mathcal{K})$ is a unital $*$-representation of $\mathcal{A}$ on some Hilbert space $\mathcal{K}$, and $V$ is a bounded operator from $\mathcal{H}$ to $\mathcal{K}$. 

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Generators of QMS
Let $L$ be the generator of a uniformly continuous QMS on $B(\mathcal{H})$. Then there exists an operator $G \in B(\mathcal{H})$, a unital $\ast$-representation of $B(\mathcal{H})$ on some Hilbert space $\mathcal{K}$, and a $V \in B(\mathcal{K}, \mathcal{H})$ such that

$$L(A) = V^* \pi(A) V + GA + AG^*$$

for all $A \in B(\mathcal{H})$.

Note: Due to a result of Kraus ('70), there exists a sequence $(V_j)_{j \geq 1} \subseteq B(\mathcal{K}, \mathcal{H})$ such that

$$V^* \pi(A) V = \sum_{j=1}^{\infty} V_j^* AV_j$$

where the series $\sum_{j=1}^{\infty} V_j^* AV_j$ converge strongly.
Question

Does the generator of a general QMS (that is, one which is not uniformly continuous) have a similar form?

Note 1: Many important examples of QMS are not uniformly continuous (for example, the QMS associated to the noncommutative heat equation).

Note 2: The QMS $(T_t)_{t \geq 0}$ is uniformly continuous if and only if $L$ is bounded.
Davies ('79)

Let $T_t : S_1(\mathcal{H}) \to S_1(\mathcal{H})$ be a semigroup which satisfies:

- $T_t^*(C(\mathcal{H})) \subseteq C(\mathcal{H})$ for all $t \geq 0$,
- There exists $e \in \mathcal{H}\setminus\{0\}$ such that $T_t(|e\rangle\langle e|) = |e\rangle\langle e|$, and
- the map $[0, \infty) \ni t \mapsto T_t(A) \in \mathcal{B}(\mathcal{H})$ is SOT-continuous for all $A \in \mathcal{B}(\mathcal{H})$.

Then there exists a dense linear subspace $D$ of $\mathcal{H}$ and linear operators $G : D \to \mathcal{H}$ and $L_n : D \to \mathcal{H}$ such that the infinitesimal generator $L$ of $(T_t)_{t \geq 0}$ is given by

$$L(A) = \sum_{n=1}^{\infty} L_n AL_n^* + GA + AG^*$$

for all $A \in (G - 1)^{-1}S_1(\mathcal{H})(G^* - 1)^{-1}$. 
Holevo (‘95)

Let $(T_t)_{t \geq 0}$ be a QMS on $\mathcal{B}(\mathcal{H})$. Assume that there exists a dense linear subspace $D$ of $\mathcal{H}$ such that

$$\lim_{t \to 0} \left\langle x, \frac{T_t A - A}{t} y \right\rangle$$

exists for all $A \in \mathcal{B}(\mathcal{H})$ and all $x, y \in D$. Then there exists a linear operator $G : D \to \mathcal{H}$, a separable Hilbert space $\mathcal{H}_0$, and a linear operator $\mathcal{L} : D \to \mathcal{H} \otimes \mathcal{H}_0$ such that

$$\langle x, \mathcal{L}(A)y \rangle = \langle \mathcal{L}x, (A \otimes 1_0)(\mathcal{L}y) \rangle_{\mathcal{H} \otimes \mathcal{H}_0} + \langle Gx, Ay \rangle + \langle x, AGy \rangle$$

for all $A \in \mathcal{B}(\mathcal{H})$ and all $x, y \in D$. 
Notation

If $\mathcal{H}$ is a Hilbert space and $D$ is a linear subspace of $\mathcal{H}$, let $S(D)$ denote the set of sesquilinear forms on $D \times D$.

Definition

Let $D$ be a linear subspace of $\mathcal{H}$ and $A$ be a linear subspace of $\mathcal{B}(\mathcal{H})$. A linear map $\phi : A \rightarrow S(D)$ is called D-completely positive if for any $k \in \mathbb{N}$, and any positive operator $A = (A_{i,j})_{1 \leq i,j \leq k} \in A \otimes M_k(\mathbb{C})$ and for all $x_1, \ldots, x_k \in D$,

$$\sum_{i,j=1}^{k} \phi(A_{i,j})(x_i, x_j) \geq 0.$$
Definition

Let \((T_t)_{t \geq 0}\) be a QDS, \(L\) be its generator and \(\text{Dom}(L)\) its domain. Then

\[ \mathcal{A} = \{ A \in \text{Dom}(L) : A^*A, AA^* \in \text{Dom}(L) \} \]

is the **domain algebra** and is equal to the largest \(^*\)-subalgebra of \(\text{Dom}(L)\) by Arveson ('02).
Androulakis, Z. (‘15)

Let \( L \) be the infinitesimal generator of a QMS on \( \mathcal{B}(\mathcal{H}) \) and let \( \mathcal{A} \) be its domain algebra. Assume that there exists \( e \in \mathcal{H} \) such that \( |e\rangle \langle e| \in \text{Dom}(L) \). Let

\[
\mathcal{D}_e = \{ x \in \mathcal{H} : |x\rangle \langle e| \in \mathcal{A} \}.
\]

Then there exists a linear map \( G : \mathcal{D}_e \to \mathcal{H} \) and a \( \mathcal{D}_e \)-completely positive map \( \phi : \mathcal{A} \to S(\mathcal{D}_e) \) such that

\[
\langle x, L(A) \rangle = \phi(A)(x, y) + \langle x, GAy \rangle + \langle GA^*x, y \rangle
\]

for all \( A \in \mathcal{A} \) and \( x, y \in \mathcal{D}_e \).
Androulakis, Z. (‘15)

Let $\mathcal{A}$ be a unital $*$-subalgebra of $B(\mathcal{H})$, $D$ be a linear subspace of $\mathcal{H}$, and $\phi : \mathcal{A} \to S(D)$ be a $D$-completely positive map. Then there exists a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : \mathcal{A} \to B(\mathcal{K})$ and a linear map $V : D \to \mathcal{K}$ such that

$$\phi(A)(x, y) = \langle Vx, \pi(A)Vy \rangle_{\mathcal{K}}$$

for all $x, y \in D$. 
Corollary

Let $L$ be the infinitesimal generator of a QMS on $\mathcal{B}(\mathcal{H})$ and let $\mathcal{A}$ be its domain algebra. Assume that there exists $e \in \mathcal{H}$ such that $|e\rangle\langle e| \in \text{Dom}(L)$. Let

$$D_e = \{ x \in \mathcal{H} : |x\rangle\langle e| \in \mathcal{A} \}.$$

Then there exists a Hilbert space $\mathcal{K}$, a $\ast$-representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$, and a linear map $V : D \rightarrow \mathcal{K}$ such that

$$\langle x, L(A)y \rangle = \langle Vx, \pi(A)Vy \rangle_{\mathcal{K}} + \langle x, GAy \rangle + \langle GA^* x, y \rangle$$

for all $A \in \mathcal{A}$ and $x, y \in D_e$.

Note: Can take $G$ to be

$$G(x) = L(|x\rangle\langle e|)e - \frac{1}{2} \langle e, L(|e\rangle\langle e|)e \rangle x.$$
Example 1

(Parthasarathy (‘92)). Let \((B_t)_{t \geq 0}\) be standard Brownian motion, \(V\) be a selfadjoint operator on \(\mathcal{H}\) and define \(T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\) by

\[ T_t(A) = \mathbb{E}[e^{iB_t V} A e^{-iB_t V}]. \]

Then \((T_t)_{t \geq 0}\) is a QMS.

If \(\mathcal{H} = L^2(\mathbb{R})\), \(V = i \frac{d}{dx}\), \(e(t) = e^{-t^2/2}\) then \(D_e\) is dense in \(L^2(\mathbb{R})\) and

\[ \langle x, L(A)y \rangle = \langle Vx, AVy \rangle + \langle x, -\frac{1}{2} V^2 Ay \rangle + \langle -\frac{1}{2} V^2 A^* x, y \rangle \]

for all \(x, y \in U_e\) and \(A \in \mathcal{A}\).
Example 2

(Fagnola (‘00), Arveson, (‘02)). Let $\mathcal{H} = L^2[0, \infty)$, and let $U_t : \mathcal{H} \to \mathcal{H}$ be defined by

$$U_t(g)(s) = \begin{cases} g(s - t) & \text{if } s \geq t \\ 0 & \text{otherwise} \end{cases}$$

Define $E_t : L^2[0, \infty) \to L^2[0, t)$ the natural projection.

Define $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by $\omega(A) = \langle f, Af \rangle$ where $f \in \mathcal{H}$ is defined by $f(s) = e^{-s}$ ($s \in [0, \infty)$).

Define $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$T_t(A) = \omega(A)E_t + U_tAU_t^*.$$ 

Then $(T_t)_{t \geq 0}$ is a QMS.
Fix $e \in L^2[0, \infty)$ such that $\mathcal{D}e \in L^2[0, \infty)$ (where $\mathcal{D}$ is the differentiation operator), and $\langle e, f \rangle = 0$.
Then $D_e \subseteq \{x \in L^2[0, \infty) : \langle x, f \rangle = 0\}$ hence $D_e$ is not dense in $\mathcal{H}$.
Also $\mathcal{A}$ is not SOT dense in $\mathcal{B}(\mathcal{H})$.
We have
\[
\langle x, L(A)y \rangle = \omega(A)x(0)y(0) + \langle x, \mathcal{D}Ay \rangle + \langle \mathcal{D}A^*x, y \rangle
\]
for all $x, y \in D_e$ and $A \in \mathcal{A}$. 

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Thank you!