ON THE ASYMPTOTIC DISTRIBUTION OF PARAMETERS IN RANDOM WEIGHTED STAIRCASE TABLEAUX

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ABSTRACT. In this paper, we study staircase tableaux, a combinatorial object introduced due to its connections with the asymmetric exclusion process (ASEP) and Askey-Wilson polynomials. Due to their interesting connections, staircase tableaux have been the object of study in many recent papers. More specific to this paper, the distribution of various parameters in random staircase tableaux has been studied. There have been interesting results on parameters along the main diagonal, however, no such results have appeared for other diagonals. It was conjectured that the distribution of the number of symbols along the kth diagonal is asymptotically Poisson as k and the size of the tableau tend to infinity. We partially prove this conjecture; more specifically we prove it for the second and the third main diagonal.

1. INTRODUCTION

In this paper, we study staircase tableaux, a combinatorial object introduced (in [10], [11]) due to connections with the asymmetric exclusion process (ASEP) and Askey-Wilson polynomials. The ASEP can be defined as a Markov chain with n sites, with at most one particle occupying each site. Particles may jump to any neighboring empty site with rate u to the right and rate q to the left. Particles may enter and exit at the first site with rates α and γ respectively. Similarly, particles may enter and exit the last site with rates δ and β. The ASEP is an interesting particle model that has been studied extensively in mathematics and physics. It has also been studied in many other fields, including computational biology [3], and biochemistry, specifically as a primitive model for protein synthesis [17]. Staircase tableaux were introduced per a connection between the steady state distribution of the ASEP and the generating function for staircase tableaux [11]. See Section 2 for a discussion of the ASEP and its connection with staircase tableaux.

In addition to interest in its own right, the ASEP has been known to have interesting connections in combinatorics and analysis. Consequently, staircase tableaux have similarly been connected to many combinatorial objects and a family of polynomials. In fact, the generating function for staircase tableaux has been used to give a formula for the moments of Askey-Wilson polynomials [11], [7]. Staircase tableaux have also inherited many interesting properties from other types of tableaux (See [1], [5], [6], [11], [8], [9], [15], [18]). We refer to [14, Table 1] for a description of some of the bijections between the various types of the tableaux.

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Due to all these interesting connections, staircase tableaux have been the object of study in many recent papers. In particular, the asymptotic distribution of parameters along the main diagonal is known. The number of α/γ symbols and the number of β/δ symbols along the main diagonal were proven to be asymptotically normal in [12], and the distribution of boxes along the main diagonal was given in [14]. However, the distributions of parameters on the other diagonals have not been studied specifically.

In [14], the distribution of each box in a staircase tableau was given, and the expected values of the number of symbols on the diagonals were computed. Based on that it was conjectured that the distribution of the number of symbols along the kth diagonal is asymptotically Poisson as k and the size of the tableau tend to infinity. The main results of this paper is the proof of this in two special cases when $k = n - 1$ and $k = n - 2$. That is, we show that the distribution of the number of symbols along the second and the third main diagonal is asymptotically Poisson with parameter 1 (see Theorem 9 and Theorem 15 below). Similarly, we show that the number of α's (resp. β's) along the second and the third main diagonal is asymptotically Poisson with parameter 1/2 (see Theorem 6, Corollary 7, and Theorem 12).

The paper is organized as follows. We introduce terminology, notation and provide some background information in the next section. Section 3 concerns the second main diagonal and the last section is devoted to the third main diagonal. The results for the second main diagonal were discussed in an extended abstract [16].

2. Notation and Preliminaries

Staircase tableaux were first introduced in [10] and [11] as follows:

Definition 1. *A staircase tableau of size n is a Young diagram of shape (n, n-1, ..., 1) such that:*

1. *The boxes are empty or contain an α, β, γ, or δ.*
2. *All boxes in the same column and above an α or γ are empty.*
3. *All boxes in the same row and to the left of an β or δ are empty.*
4. *Every box on the diagonal contains a symbol.*

The rows and columns in a staircase tableau are numbered from 1 through n, beginning with the box in the NW-corner and continuing south and east respectively. Each box is numbered (i, j) where $i, j \in \{1, 2, ..., n\}$. Note that $i+j \leq n+1$. We refer to the collection of boxes $(n-i+1, i)$ such that $i = 1, 2, ..., n$ as the main diagonal, the collection of boxes $(n-i, i)$ such that $i = 1, 2, ..., n-1$ as the second main diagonal, and the collection of boxes $(n-i-1, i)$ such that $i = 1, 2, ..., n-2$ as the third main diagonal.

Following the conventions of [14], $\mathcal{S}_n$ is the set of all staircase tableaux of size n. For a given $S \in \mathcal{S}_n$, the number of α's, γ's, β's and δ's in S are denoted by $N_\alpha, N_\beta, N_\gamma$, and N_δ respectively. The weight of S is the product of all symbols in S:

$$wt(S) = \alpha^{N_\alpha}\beta^{N_\beta}\gamma^{N_\gamma}\delta^{N_\delta}.$$

It was known (see e.g. [4]) that the generating function $Z_n(\alpha,\beta,\gamma,\delta) := \sum_{S\in\mathcal{S}_n} wt(S)$ is equal to the product:

$$Z_n(\alpha,\beta,\gamma,\delta) = \prod_{i=0}^{n-1}(\alpha+\beta+\delta+\gamma+i(\alpha+\gamma)(\beta+\delta)). \tag{1}$$

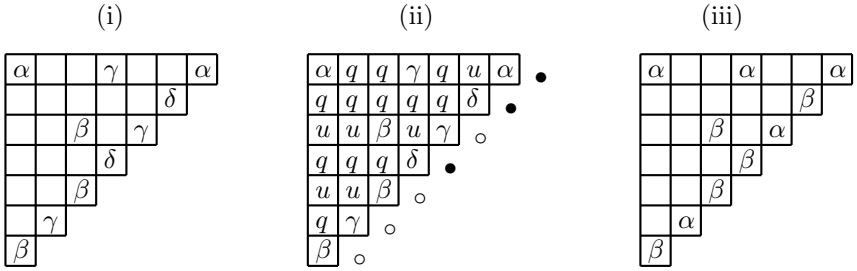

FIGURE 1. A staircase tableau of size 7 with weight $\alpha^2\beta^3\delta^2\gamma^3$. (ii) The extension of (i) to a staircase tableau of weight $\alpha^2\beta^3\delta^2\gamma^3u^6q^{13}$ and type $\bullet\bullet\circ\bullet\circ\circ\circ$. (iii) The α/β-staircase tableau obtained from (i) by replacing γ's with α's and δ's with β's. It's weight is $\alpha^5\beta^5$.

Notice that there is an involution on the staircase tableaux of a given size obtained by interchanging the rows and the columns, α's and β's, and γ's and δ's, see further [4]. In particular, the fact that α's and β's are identical up to this involution allows us to extend results for α's to results for β's.

The connection between staircase tableaux and the ASEP requires an extension of this preceding definition. After following the rules from Definition 1, we then fill all the empty boxes with u's and q's, the rates at which particles in the ASEP jump to the right and left respectively. We do so by first filling all boxes to the left of a β with a u and to the left of a δ with a q. Then, we fill the empty boxes with a u if it is above an α or a δ, and q otherwise. The weight of a staircase tableau filled as such is defined in the same way, the product of the parameters in each box. Also, the total weight of all such staircase tableaux, which we denote by $\mathcal{S}_n^{'}$, is given by:

$$Z_n(\alpha,\beta,\gamma,\delta,q,u) := \sum_{S\in\mathcal{S}_n'} wt(S).$$

Then, each staircase tableau of size n is associated with a state of the ASEP with n sites (See Figure 1). This is done by aligning the Markov chain with the diagonal entries of the staircase tableau. A site is filled if the corresponding diagonal entry is an α or a γ and a site is empty if the corresponding diagonal entry is a β or a δ. Each staircase tableau's associated state of the ASEP is called its type.

Using this association, it was shown in [11] that the steady state probability that the ASEP is in state η is given by:

$$\frac{\sum_{T\in\mathfrak{T}} wt(T)}{Z_n},$$

where $\mathfrak{T}$ is the set of all staircase tableau of type η.

For the purposes of this paper, we will consider more simplified staircase tableaux, namely α/β-staircase tableaux as introduced in [14], which are staircase tableaux limited to the symbols α and β. The set $\overline{\mathcal{S}}_n \subset \mathcal{S}_n$ denotes the set of all such staircase tableaux. Since the symbols α and γ follow the same rules in the definition, as do β and δ, any $S \in \mathcal{S}_n$ can be obtained from an $S' \in \overline{\mathcal{S}}_n$ by replacing the appropriate α's with γ's and β's with δ's.

The generating function of α/β-staircase tableaux is:

$$Z_n(\alpha,\beta) := \sum_{S\in\overline{\mathcal{S}}_n} wt(S) = Z_n(\alpha,\beta,0,0)$$

and it follows from (1) that it is simply:

$$Z_n(\alpha,\beta) = \alpha^n\beta^n(a+b)^{\overline{n}} = \alpha^n\beta^n(a+b+n-1)_n$$

where $a := \alpha^{-1}$ and $b := \beta^{-1}$, a notation that will be used frequently throughout this paper, $(x)^{\overline{n}}$ is the rising factorial of x, i.e. $(x)^{\overline{n}} = x(x+1)\cdots(x+n-1)$, and $(x)_n = x(x-1)\dots(x-(n-1))$ is the falling factorial of x.

We wish to consider random staircase tableaux as was done in [12] but it suffices to study random α/β-staircase tableaux as was done in [14]. All of our results for random α/β-staircase tableaux can be extended to random staircase tableaux with all four parameters, $\alpha,\gamma,\beta,\delta$. This is done by randomly replacing each α with γ with probability $\frac{\gamma}{\alpha+\gamma}$ and similarly, each β with δ with probability $\frac{\delta}{\beta+\delta}$ independently for each occurrence. Notice that $Z_n(\alpha,\beta,\gamma,\delta) = Z_n(\alpha+\gamma,\beta+\delta)$. We also allow all parameters to be arbitrary positive real numbers, i.e. $\alpha,\beta \in (0,\infty)$, allowing $\alpha = \infty$ by fixing β and taking the limit or vice versa, or $\alpha = \beta = \infty$ by taking the limit. Several such cases were considered in the literature and we refer to [14, Section 3] for examples and discussion.

The following is a formal definition of a weighted random staircase introduced in [14], although we prefer to use a different notation:

Definition 2. *For all $n \geq 1$, $\alpha,\beta \in [0,\infty)$ with $(\alpha,\beta) \neq (0,0)$, we consider a family of probability measures $\mathbb{P}_{n,\alpha,\beta}$ on $\overline{\mathcal{S}}_n$ defined by:*

$$\mathbb{P}_{n,\alpha,\beta}(S) = \frac{wt(S)}{Z_n(\alpha,\beta)} = \frac{\alpha^{N_\alpha}\beta^{N_\beta}}{Z_n(\alpha,\beta)}, \quad S \in \overline{\mathcal{S}}.$$

We denote by $\overline{\mathcal{S}}_{n,\alpha,\beta}$ the probability space $(\overline{\mathcal{S}}_n, \mathbb{P}_{n,\alpha,\beta})$ and we call $S \in \overline{\mathcal{S}}_{n,\alpha,\beta}$ a random weighted staircase tableau (with weights α, β). That is, S is an α/β-staircase tableau in $\overline{\mathcal{S}}_n$ chosen according to the probability distribution $\mathbb{P}_{n,\alpha,\beta}$.

As in [14] α and β are used in two different meanings, as fixed symbols in the tableaux and as the values of the parameters, but that should not cause any confusion.

We identify the boxes of a tableau by a row and column number (as in a matrix) and we write $y_{i,j}$ to indicate that a box (i,j) contains a symbol y, where y is α, β, or 0. Using the above definition, Hitczenko and Janson presented the distribution of a given box in a random staircase tableau. If a box is on the main diagonal, its distribution, in our notation, is (see [14, Theorem 7.1]):

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{i,n+1-i}) = \frac{n-i+b}{n+a+b-1}, \quad \mathbb{P}_{n,\alpha,\beta}(\alpha_{i,n+1-i}) = \frac{n-i+b}{n+a+b-1}. \tag{2}$$

Since a box on the main diagonal is never empty, the second formula follows trivially from the first. This is no longer true for boxes not on the main diagonal. For such boxes the distribution of non–zero symbols is (see [14, Theorem 7.2]):

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{i,j}) = \frac{j-1+b}{(i+j+a+b-1)_2}, \qquad \mathbb{P}_{n,\alpha,\beta}(\beta_{i,j}) = \frac{i-1+a}{(i+j+a+b-1)_2}, \tag{3}$$

and the remainder is for $\mathbb{P}_{n,\alpha,\beta}(0_{i,j})$.

For an arbitrary $S \in \overline{\mathcal{S}}_n$ and an arbitrary box (i,j) in S, define $S[i,j]$ to be the subtableau in $\overline{\mathcal{S}}_{n-i-j+2}$ obtained by deleting the first $i-1$ rows and $j-1$ columns, see [14]. The following statement was proven in [14, Theorem 6.1] and is a useful tool in our results: If $S \in \overline{\mathcal{S}}_{n,\alpha,\beta}$ then $S[i,j] \in \overline{\mathcal{S}}_{n-i-j+2,\widehat{\alpha},\widehat{\beta}}$ with $\widehat{a} = a+i-1$ and $\widehat{b} = b+j-1$. This means that if $\widetilde{S} \in \overline{\mathcal{S}}_{n-i-j+2}$ then

$$\mathbb{P}_{n,\alpha,\beta}(S \in \overline{\mathcal{S}}_n : \ S[i,j] = \widetilde{S}) = \mathbb{P}_{n-i-j+2,\widehat{\alpha},\widehat{\beta}}(\widetilde{S}), \tag{4}$$

To show the convergence to a Poisson random variable we will rely on the method of (factorial) moments. When the random variables are the sums of indicators the condition takes the following form

Lemma 1. *Let $Y = \sum_{j=1}^m I_j$, where (I_j) are indicator random variables. Then, for $r \geq 1$,*

$$\mathbb{E}(Y)_r = r! \left(\sum_{1 \leq j_1 < \ldots < j_r \leq m} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}) \right),$$

where $\mathbb{E}X)_r = \mathbb{E}X(X-1)\ldots(X-(r-1))$ is the r^{th} factorial moment.

Proof. We have

$$\begin{aligned} z^Y &= z^{\sum_{j=1}^m I_j} = \prod_{j=1}^m z^{I_j} = \prod_{j=1}^m (1+(z-1))^{I_j} = \prod_{j=1}^m (1+I_j(z-1)) \\ &= 1 + \sum_{r=1}^m \left(\sum_{1 \leq j_1 < \ldots < j_r \leq m} \left(\prod_{k=1}^r I_{j_k} \right) \right) (z-1)^r \\ &= 1 + \sum_{r=1}^m (z-1)^r \left(\sum_{1 \leq j_1 < \ldots < j_r \leq m} \left(\prod_{k=1}^r I_{j_k} \right) \right). \end{aligned}$$

Thus,

$$\mathbb{E}(z^Y) = 1 + \sum_{r=1}^m (z-1)^r \left(\sum_{1 \leq j_1 < \ldots < j_r \leq m} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}) \right).$$

Hence

$$\mathbb{E}(Y)_r = \frac{d^r}{dz^r}(\mathbb{E}z^Y)|_{z=1} = r! \left(\sum_{1 \leq j_1 < \ldots < j_r \leq m} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}) \right).$$

□

Thus, to apply the above lemma we will need to find the joint distribution of symbols in given boxes on the diagonal of a tableau. We will obtain the exact formulas for the second main diagonal (see Theorems 4 and 8 in the next section)

and asymptotic for the third main diagonal (see Lemmas 11 and 14 below). It should be noted that the joint distribution of symbols for the main diagonal was given in [14, Theorem 7.6], but since boxes on the main diagonal cannot be empty it is a different issue.

3. Distribution of parameters along the second main diagonal

The following two lemmas consider the probability of an arbitrary staircase tableau in $\overline{\mathcal{S}}_n$ that is conditioned on having an α or a β in the box $(n-1,1)$. The statements follow almost immediately from the definition of a staircase tableau, but will be used frequently throughout the paper.

Lemma 2. *If $S \in \overline{\mathcal{S}}_{n,\alpha,\beta}$ is conditioned on having α in box $(n-1,1)$, then the subtableau $S_{n,\alpha,\beta}[1,3] \in \overline{\mathcal{S}}_{n-2,\alpha,\beta}$. In other words, for $S \in \overline{\mathcal{S}}_{n,\alpha,\beta}$*

$$\mathbb{P}_{n,\alpha,\beta}(S \mid \alpha_{n-1,1}) = \mathbb{P}_{n-2,\alpha,\beta}(S[1,3]).$$

Proof. If $S \in \overline{\mathcal{S}}_n$ is a staircase tableau such that $\alpha_{n-1,1}$, then $\beta_{n,1}$ and $\alpha_{n-1,2}$ by the rules of a staircase tableau. The first and second column are otherwise empty by those same rules. The remainder, $S[1,3]$, is an arbitrary staircase tableau of size $n-2$. Therefore, the lemma follows. □

Lemma 3. *Let $(S)_{i,j}$ be a subtableau of S with the ith row and the jth column removed. If $S \in \overline{\mathcal{S}}_{n,\alpha,\beta}$ is conditioned on having β in box $(n-1,1)$, then the subtableau $(S)_{n-1,2}$ is random tableau in $\overline{\mathcal{S}}_{n-1,\alpha,\beta}$ conditioned on having a β in the $(n-1,1)$ box. In other words*

$$\mathbb{P}_{n,\alpha,\beta}(S \mid \beta_{n-1,1}) = \mathbb{P}_{n-1,\alpha,\beta}((S)_{n-1,2} \mid \beta_{n-1,1}).$$

Proof. If $S \in \overline{\mathcal{S}}_n$ is a staircase tableau such that $\beta_{n-1,1}$, then $\alpha_{n-1,2}$ and $\beta_{n-1,1}$ by the rules of a staircase tableau. The second column is otherwise empty by those same rules. The nth row only has one box, $(n,1)$, which must be a β. The remainder is an arbitrary staircase tableau of size $n-1$ conditioned to have a β in box $(n-1,1)$. Therefore, the lemma follows. □

We are now ready to state and prove the results for the second main diagonal. We begin with the distribution of the number of α's and we treat the number of non–empty boxes in the next subsection.

3.1. The number of α's on the second main diagonal.

As our first result, the following is the distribution of boxes along the second main diagonal. In order to simplify notation, let α_j be the event $\alpha_{n-j,j}$ i.e. that a box $(n-j,j)$ on the second diagonal and in the j^{th} column contains an α.

Theorem 4. *Let $1 \le j_1 < ... < j_r \le n-1$. If*

$$j_k \le j_{k+1} - 2, \quad \forall k = 1, 2, ..., r-1 \tag{5}$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n+a+b-2r+2k-1)_2}.$$

(For $r = 1$, this is (3)). Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = 0.$$

Proof. First note that when (5) fails there exists j_k such that $j_k = j_{k+1} - 1$ and thus there must be two α's in boxes side by side on the $(n-i, i)$ diagonal. But this is impossible by the rules of a staircase tableau as no symbol can be put in the diagonal box $(n - j_k, j_{k+1})$ adjacent to these two boxes. Therefore the probability is 0.

Suppose now that (5) holds. We proceed by induction on r. Set

$$\widehat{\beta}^{-1} = \beta^{-1} + j_1 - 1, \quad \widehat{n} := n - (j_1 - 1), \quad \widehat{j}_l := j_l - (j_1 - 1), \quad 1 \le l \le r. \tag{6}$$

Then by (4)

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_1}, \ldots, \alpha_{\widehat{j}_r}) = \mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_2}, \ldots, \alpha_{\widehat{j}_r} | \alpha_{\widehat{j}_1}) \cdot \mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_1}).$$

Since $\widehat{j}_1 = 1$, by Lemma 2 and the induction hypothesis (applied with $\widetilde{n} := \widehat{n} - 2$, $\widetilde{j}_l := \widehat{j}_{l+1} - 2$, $1 \le l \le r-1$)

$$\begin{aligned}
&\mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_2}, \ldots, \alpha_{\widehat{j}_r} \mid \widehat{\alpha}_{j_1}) = \mathbb{P}_{\widehat{n}-2,\alpha,\widehat{\beta}}(\alpha_{\widehat{j}_2-2}, \ldots, \alpha_{\widehat{j}_r-2}) \\
&= \prod_{k=1}^{r-1} \frac{\widehat{b} + \widetilde{j}_{(r-1)-k+1} - 2(r-1) + 2k - 1}{(\widetilde{n} + a + \widehat{b} - 2(r-1) + 2k - 1)_2} = \prod_{k=1}^{r-1} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n + a + b - 2r + 2k - 1)_2},
\end{aligned}$$

where, in the last step we used (6). By (3) and (6):

$$\mathbb{P}_{\widehat{n},\alpha,\widehat{\beta}}(\alpha_1) = \frac{\widehat{b}}{(\widehat{n} + a + \widehat{b} - 1)_2} = \frac{b + j_1 - 1}{(n + a + b - 1)_2}.$$

Therefore,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \ldots, \alpha_{j_r}) \quad = \quad \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n + a + b - 2r + 2k - 1)_2}$$

which proves the result. □

Our second main result of this section is the distribution of the number of α's (and β's) along the second main diagonal. The proof requires a lemma.

Lemma 5. *Let*

$$J_{r,m} := \{1 \le j_1 < \ldots < j_r \le m : \ j_k \le j_{k+1} - 2, \ \ \forall k = 1, 2, \ldots, r-1\}.$$

Then

$$\sum_{J_{r,m}} \left(\prod_{k=1}^{r} j_{r-k+1} \right) = \frac{(m+1)_{2r}}{2^r r!}.$$

Proof. By induction on r. When $r = 1$:

$$\sum_{J_{1,m}} \left(\prod_{k=1}^{1} j_{1-k+1} \right) = \sum_{j_1=1}^{m} j_1 = \frac{(m+1)m}{2}.$$

Assume the statement holds for $r-1$. Then:

$$\begin{aligned}\sum_{J_{r,m}}\left(\prod_{k=1}^{r} j_{r-k+1}\right) &= \sum_{j_r=2r-1}^{m} j_r\left(\sum_{J_{r-1,j_r-2}}\prod_{k=2}^{r} j_{r-k+1}\right)\\ &= \sum_{j_r=2r-1}^{m} j_r\frac{(j_r-1)_{2(r-1)}}{2^{r-1}(r-1)!}\\ &= \frac{1}{2^{r-1}(r-1)!}\sum_{j_r=2r-1}^{m}(j_r)_{2r-1}\end{aligned}$$

where the second equality is by the induction hypothesis. Since

$$\sum_{j_r=2r-1}^{m}(j_r)_{2r-1}=\sum_{j_r=0}^{m}(j_r)_{2r-1}$$

the lemma will be proved once we verify that

$$\sum_{j=0}^{m}(j)_t=\frac{(m+1)_{t+1}}{t+1},$$

for any non-negative integer t (and apply it with $t=2r-1$). Using the identity

$$\sum_{j=0}^{m}\binom{j}{t}=\binom{m+1}{t+1}$$

(see, e.g. [13, Formula (5.10)]) we see that

$$\sum_{j=0}^{m}(j)_t=\sum_{j=0}^{m}\frac{j!}{(j-t)!}=t!\sum_{j=0}^{m}\binom{j}{t}=t!\binom{m+1}{t+1}=t!\frac{(m+1)_{t+1}}{(t+1)!}=\frac{(m+1)_{t+1}}{m+1},$$

as asserted. □

Finally, define A_n and B_n to be the number of α's and β's on the second main diagonal, i.e. $A_n:=\sum_{j=1}^{n-1}I_{\alpha_j}$ and $B_n:=\sum_{j=1}^{n-1}I_{\beta_j}$, where β_j means $\beta_{n-j,j}$. Then, the asymptotic distribution of A_n and B_n is given in the following theorem and corollary.

Theorem 6. *Let $Pois(\lambda)$ be a Poisson random variable with parameter λ. Then, as $n\to\infty$,*

$$A_n \xrightarrow{d} Pois\left(\frac{1}{2}\right). \tag{7}$$

Proof. By [2, Theorem 20, Chapter 1] it suffices to show that the rth factorial moment of A_n satisfies:

$$\mathbb{E}(A_n)_r\to\left(\frac{1}{2}\right)^r \quad \text{as } \to\infty. \tag{8}$$

By Lemma 1, Theorem 4, and Lemma 5

$$\begin{aligned}\mathbb{E}(A_n)_r &= r!\sum_{J_{r,n-1}}\left(\prod_{k=1}^{r}\frac{b+j_{r-k+1}-2r+2k-1}{(n+a+b-2r+2k-1)_2}\right)\approx r!\sum_{J_{r,n-1}}\left(\prod_{k=1}^{r}\frac{j_{r-k+1}}{n^2}\right)\\ &= \frac{r!}{n^{2r}}\frac{(n)_{2r}}{2^r r!}\to\left(\frac{1}{2}\right)^r, \quad \text{as } n\to\infty.\end{aligned}$$

□

Corollary 7. *The rth factorial moment of the number B_n of β's on the second main diagonal of a random staircase tableau of size n satisfies:*

$$\mathbb{E}(B_n)_r \to \left(\frac{1}{2}\right)^r \quad \text{as } n \to \infty. \tag{9}$$

Furthermore,

$$B_n \xrightarrow{d} Pois\left(\frac{1}{2}\right) \quad \text{as } n \to \infty. \tag{10}$$

Proof. This follows by symmetry, see Section 1. □

Remark 1. *Theorem 6 and Corollary 7 hold regardless of the values of α and β including the cases discussed earlier when $\alpha = \infty$, $\beta = \infty$, or $\alpha = \beta = \infty$. As noted in* [14, Examples 3.6 and 3.7] *these cases correspond to staircase tableaux with the maximal number of α's (or β's) and the maximal number of symbols, respectively. The same applies to Theorems 9, 11, and 15 below.*

3.2. **Distribution of non-empty boxes on the second main diagonal.** The number of α's and the number of β's, N_a and N_b, are not independent random variables, and the second main diagonal may have empty boxes. Therefore, in order to completely describe the second main diagonal, we must consider both symbols collectively. First, we present the distribution of non-empty boxes along the second main diagonal. We let x_j denote the event that the box $(n-j, j)$ in the jth column and on the second main diagonal in non–empty.

Theorem 8. *Let $1 \le j_1 < ... < j_r \le n-1$. If (5) holds then*

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_r}) = \prod_{k=1}^{r} \frac{1}{n+a+b-r+k-1}.$$

(For $r = 1$, this is obtained by adding the expressions in (3).)Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_r}) = 0.$$

Proof. Suppose (5) holds. We proceed by induction on r. As in the proof of Theorem 4 by passing to $\widehat{n} := n - (j_1 - 1)$, $\widehat{j_i} := j_i - (j_1 - 1)$, and $\widehat{b} := b + j_1 - 1$ we may assume that $j_1 = 1$.

By the law of total probability we have

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1, \ldots, x_{j_r}) &= \mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_1 = \alpha)\mathbb{P}_{n,\alpha,\beta}(x_1 = \alpha) \\ &\quad + \mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_1 = \beta)\mathbb{P}_{n,\alpha,\beta}(x_1 = \beta).\end{aligned}$$

Now consider two cases:

Case 1: $x_1 = \alpha$. By Lemma 2 and the induction hypothesis:

$$\begin{aligned}&\mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_1 = \alpha) = \mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2}, \ldots, x_{j_r-2}) \\ &= \prod_{k=1}^{r-1} \frac{1}{n-2+a+b-(r-1)+k-1} = \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-2}\end{aligned}$$

and by (3),

$$\mathbb{P}_{n,\alpha,\beta}(x_1 = \alpha) = \frac{b}{(n+a+b-1)_2}.$$

Therefore,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_r}) = \frac{b}{(n+a+b-1)_2} \cdot \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-2}.$$

Case 2: $x_1 = \beta$. By Lemma 3

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_{j_2}, \ldots, x_{j_r} \mid x_{j_1} = \beta) &= \mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1} \mid \beta_{n-1,1}) \\ &= \frac{\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1}, \beta_{n-1,1})}{\mathbb{P}_{n-1,\alpha,\beta}(\beta_{n-1,1})}.\end{aligned}$$

The numerator is equal to

$$\begin{aligned}&\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1}) - \mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1},\ \alpha_{n-1,1}) \\ &\quad = \mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1}) \\ &\qquad -\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1} \mid \alpha_{n-1,1})\mathbb{P}_{n-1,\alpha,\beta}(\alpha_{n-1,1}).\end{aligned} \tag{11}$$

By [14, Lemma 7.5] and the induction hypothesis the conditional probability above is

$$\mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2}, \ldots, x_{j_r-2}) = \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-2}. \tag{12}$$

By (3), (2), and the induction hypothesis,

$$\mathbb{P}_{n,\alpha,\beta}(x_1 = \beta) = \frac{n+a-2}{(n+a+b-1)_2} \tag{13}$$

$$\frac{1}{\mathbb{P}_{n-1,\alpha,\beta}(\beta_{n-1,1})} = \frac{n+a+b-2}{n+a-2} \tag{14}$$

$$\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1}, \ldots, x_{j_r-1}) = \prod_{k=1}^{r-1} \frac{1}{n-1+a+b-r+k} \tag{15}$$

$$\mathbb{P}_{n-1,\alpha,\beta}(\alpha_{n-1,1}) = \frac{b}{n+a+b-2}. \tag{16}$$

Combining (12) - (16):

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_r}) = \frac{1}{n+a+b-1} \cdot \Big(&\prod_{k=1}^{r-1} \frac{1}{n-1+a+b-r+k} \\ &- \frac{b}{n+a+b-2} \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-2} \Big).\end{aligned}$$

Adding Case 1 and Case 2:

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1, ..., x_{j_r}) &= \frac{1}{n+a+b-1} \cdot \prod_{k=1}^{r-1} \frac{1}{n+a+b-r+k-1} \\ &= \prod_{k=1}^{r} \frac{1}{n+a+b-r+k-1}\end{aligned}$$

which proves our assertion when (5) holds.

If there exists j_k such that $j_k = j_{k+1} - 1$, then $\{x_{j_1}, \ldots, x_{j_r}\}$ implies that two boxes side by side on the $(n-i, i)$ diagonal are non-empty, which is impossible by the rules of a staircase tableau. Therefore the probability is 0. □

As our final result of this section, we consider the number of symbols on the second main diagonal, which we denote by X_n. Then $X_n = \sum_{j=1}^{n-1} I_{x_j}$. The asymptotic distribution of the number of symbols on the second main diagonal is given by:

Theorem 9. *As $n \to \infty$,*

$$X_n \xrightarrow{d} Pois\,(1)\,. \tag{17}$$

Proof. By Theorem 8 and Lemma 1

$$\begin{aligned}
\mathbb{E}(X_n)_r &= r!|J_{r,n-1}| \prod_{k=1}^{r} \frac{1}{n+a+b-r+k-1} \\
&= r! \left(\binom{n-1}{r} + O(n^{r-1}) \right) \prod_{k=1}^{r} \frac{1}{n+a+b-r+k-1} \\
&\approx r! \cdot \frac{(n-1)^r}{r! n^r} \to 1 \quad \text{as } n \to \infty.
\end{aligned}$$

The result follows by [2, Theorem 20], as discussed in the proof of Theorem 6. □

4. The distribution of parameters on the third main diagonal

For the considerations of the third main diagonal, note that if the box $(n-k-1, k)$ is non-empty, then by the rules of staircase tableaux, the corresponding diagonal boxes $(n-k+1, k)$ and $(n-k-1, k+2)$ are non-empty (in fact, they contain β and α, respectively). The next lemma shows that the remaining two boxes $(n-k-1, k+1)$ and $(n-k, k)$ are likely to be empty. We write $x_{i,j}$ to indicate that the (i, j) box of a tableau is non–empty.

Lemma 10. *Consider arbitrary $S \in \overline{\mathcal{S}}_n$ such that $x_{n-k-1,k}$. If $x_{n-k-1,k+1}$ or $x_{n-k,k}$, then $\mathbb{P}_{n,\alpha,\beta}(S) = O\left(\frac{1}{(n+a+b)^2}\right)$.*

Proof. Note that only one of the boxes $(n-k-1, k+1)$, $(n-k, k)$ of S may contain a non–zero symbol, and in either case, under the assumptions of the lemma, the subtableau $S[n-k-1, k]$ has two α's, two β's, one empty box, and a symbol corresponding to position $(n-k-1, k)$ in S. By (4),

$$\mathbb{P}_{n,\alpha,\beta}(S) = \mathbb{P}_{3,\widehat{\alpha},\widehat{\beta}}(S[n-k-1, k]),$$

with $\widehat{a} = a + n - k - 2$ and $\widehat{b} = b + k - 1$, and by the above observation

$$\mathbb{P}_{3,\widehat{\alpha},\widehat{\beta}}(S[n-k-1, k]) = 2\frac{(\widehat{\alpha}+\widehat{\beta})(\widehat{\alpha}\widehat{\beta})^2}{Z_3(\widehat{\alpha}, \widehat{\beta})} = \frac{2}{(\widehat{a}+\widehat{b}+1)^{\bar{2}}} = O\left(\frac{1}{(n+a+b)^2}\right).$$

□

4.1. **The asymptotic distribution of α's on the third main diagonal.** We begin by deriving the asymptotic probability for the joint distribution of α's on the third main diagonal. We let $\alpha_j^{(3)}$ be the event that there is an α in the jth column on the third main diagonal (i.e in the box $(n-j-1, j)$, $1 \le j \le n-2$) but since we will be dealing exclusively with the third main diagonal through the remainder of this section we will drop the superscript.

We can now give the asymptotic joint distribution of the α's on the third main diagonal.

Lemma 11. *Let $1 \le j_1 < ... < j_r \le n-2$. If*

$$j_l \le j_{l+1} - 3, \quad \forall l = 1, 2, ..., r-1 \tag{18}$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) = \prod_{l=1}^{r} \frac{b + j_{r-l+1} - 2r + 2l - 1}{(n+a+b-2r+2l-1)_2} + O\left(\frac{1}{(n+a+b)^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) = O\left(\frac{1}{(n+a+b)^{r}}\right).$$

Proof. The proof is by induction on r. When $r = 1$,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}) &= \mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \alpha_{n-j_1-1,j_1+1}) + \mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, \beta_{n-j_1,j_1}) \\ &\quad + \mathbb{P}_{n,\alpha,\beta}(\alpha_{n-j_1-1,j_1}, 0_{n-j_1-1,j_1+1}, 0_{n-j_1,j_1}).\end{aligned}$$

By Lemma 10, each of the first two probabilities on the right–hand side is $O(1/n^2)$. By (4) (applied with $i = n - j_1 - 1$ and $j = j_1$ so that $\widehat{a} = a + n - j_1 - 2$ and $\widehat{b} = b + j_1 - 1$) and a direct computation for the resulting tableaux of size three, the last probability is

$$\frac{\widehat{\alpha}^3\widehat{\beta}}{Z_3(\widehat{\alpha},\widehat{\beta})} + \frac{\widehat{\alpha}^2\widehat{\beta}^2}{Z_3(\widehat{\alpha},\widehat{\beta})} = \frac{\widehat{\alpha}^2\widehat{\beta}}{(\widehat{\alpha}\widehat{\beta})^2(\widehat{a}+\widehat{b}+1)^{\overline{2}}} = \frac{b+j_1-1}{(n+a+b-1)_2},$$

as required.

Assume the statement holds for integers up to $r-1$. As in the earlier proofs assume without loss that $j_1 = 1$ and consider the following two cases.

Case 1. $j_2 \ge j_1 + 3 = 4$.

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1, ..., \alpha_{j_r}) &= \mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \alpha_1, ..., \alpha_{j_r}) + \mathbb{P}_{n,\alpha,\beta}(\beta_{n-1,1}, \alpha_1, \ldots, \alpha_{j_r}) \\ &\quad + \mathbb{P}_{n,\alpha,\beta}(0_{n-1,1}, 0_{n-2,2}, \alpha_1, ..., \alpha_{j_r}).\end{aligned} \tag{19}$$

The main contribution is the last probability which can be written as follows,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_2}, ..., \alpha_{j_r} | \alpha_1, 0_{n-1,1}, 0_{n-2,2}) \mathbb{P}_{n,\alpha,\beta}(\alpha_1, 0_{n-1,1}, 0_{n-2,2}).$$

The conditional probability is equal to $\mathbb{P}_{n-2,\alpha,\beta}(\alpha_{j_2-2}, ..., \alpha_{j_r-2})$ (this is because the condition forces zeroes in the first and the third column of $S[j_1 - 1, 1]$ above its $(n-2)$nd row and the remaining $n-2$ columns are unrestricted and thus form a general tableau of size $n-2$). To compute $\mathbb{P}_{n,\alpha,\beta}(\alpha_1, 0_{n-1,1}, 0_{n-2,2})$, remove the top $n-3$ rows setting $\widehat{a} := a + n - 3$ and directly calculate that this probability is

$$(\widehat{\alpha} + \beta)\frac{\widehat{\alpha}^2\beta}{Z_3(\widehat{\alpha},\beta)} = \frac{b}{(\widehat{a}+b+1)^{\overline{2}}} = \frac{b}{(n+a+b-1)_2}.$$

So then by the induction hypothesis, if $j_l \leq j_{l+1} - 3$ for all $l = 2, \ldots, r-1$ then the last probability in (19) is

$$\left(\prod_{k=1}^{r-1} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n+a+b-2r+2k-1)_2} + O\left(\frac{1}{(n+a+b)^r}\right)\right) \frac{b}{(n+a+b-1)_2}$$
$$= \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(n+a+b-2r+2k-1)_2} + O\left(\frac{1}{(n+a+b)^{r+1}}\right).$$

(Note that in view of (6) the last equality holds regardless of whether $j_1 = 1$ or $j_1 > 1$.) On the other hand, if for some $2 \leq l \leq r-1$, $j_l > j_{l+1} - 3$ then by the induction hypothesis again and (6) (which implies that $b = O(n)$) the same expression is

$$O\left(\frac{1}{(n+a+b)^{r-1}}\right) \frac{b}{(n+a+b-1)_2} = O\left(\frac{1}{(n+a+b)^r}\right).$$

Now, returning to Equation (19), the first two probabilities can be calculated as follows,

$$\begin{aligned} &= \mathbb{P}_{n,\alpha,\beta}(\alpha_{j_2}, ..., \alpha_{j_r} | \alpha_{n-2,2}, \alpha_1) \mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \alpha_1) \\ &\quad + \mathbb{P}_{n,\alpha,\beta}(\alpha_{j_2}, ..., \alpha_{j_r} | \beta_{n-1,1}, 0_{n-2,2}, \alpha_1) \mathbb{P}_{n,\alpha,\beta}(\beta_{n-1,1}, 0_{n-2,2}, \alpha_1) \\ &= \mathbb{P}_{n-3,\alpha,\beta}(\alpha_{j_2-3}, ..., \alpha_{j_r-3}) \mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \alpha_1) \\ &\quad + \mathbb{P}_{n-3,\alpha,\beta}(\alpha_{j_2-3}, ..., \alpha_{j_r-3}) \mathbb{P}_{n,\alpha,\beta}(\beta_{n-1,1}, 0_{n-2,2}, \alpha_1). \end{aligned}$$

By the inductive hypothesis

$$\mathbb{P}_{n-3,\alpha,\beta}(\alpha_{j_2-3}, ..., \alpha_{j_r-3}) = O\left(\frac{1}{(n+a+b)^{r-1}}\right) = O\left(\frac{1}{(n+a+b)^{r-1}}\right)$$

and by Lemma 10, each of the other two probabilities above is

$$O\left(\frac{1}{(n+a+b)^2}\right).$$

Hence, this probability is

$$O\left(\frac{1}{(n+a+b)^{r-1}}\right) O\left(\frac{1}{(n+a+b)^2}\right) = O\left(\frac{1}{(n+a+b)^{r+1}}\right).$$

Case 2. $j_2 < 4$.

The case where $j_2 = 2$ is impossible by the rules of staircase tableaux and hence has probability zero. For the case where $j_2 = 3$

$$\begin{aligned} \mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) &= \mathbb{P}_{n,\alpha,\beta}(\alpha_{j_3}, ..., \alpha_{j_r}) | \alpha_1, \alpha_2) \mathbb{P}_{n,\alpha,\beta}(\alpha_1, \alpha_2) \\ &= \mathbb{P}_{n-4,\alpha,\beta}(\alpha_{j_3-4}, ..., \alpha_{j_r-4}) \mathbb{P}_{n,\alpha,\beta}(\alpha_1, \alpha_2). \end{aligned}$$

To compute $\mathbb{P}_{n,\alpha,\beta}(\alpha_1, \alpha_2)$, remove the top $n-4$ rows setting $\widehat{a} := a + n - 4$ and directly calculate that

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1, \alpha_2) = \frac{b^2}{(\widehat{a}+b)^{\overline{4}}} = O\left(\frac{b^2}{(n+a+b)^4}\right) = O\left(\frac{1}{(n+a+b)^2}\right). \tag{20}$$

By the induction hypothesis,

$$\mathbb{P}_{n-4,\alpha,\beta}(\alpha_{j_3-4}, ..., \alpha_{j_r-4}) = O\left(\frac{1}{(n+a+b)^{r-2}}\right) = O\left(\frac{1}{(n+a+b)^{r-2}}\right).$$

Therefore,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) = O\left(\frac{1}{(n+a+b)^{r-2}}\right) O\left(\frac{1}{(n+a+b)^2}\right) = O\left(\frac{1}{(n+a+b)^r}\right).$$

Combining Cases 1 and 2, if $j_l \le j_{l+1} - 3, \ \ \forall\ l = 1, 2, \ldots, r-1$, then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1},...,\alpha_{j_r}) \quad = \quad \prod_{l=1}^{r} \frac{b + j_{r-l+1} - 2r + 2l - 1}{(n+a+b-2r+2l-1)_2} + O\left(\frac{1}{(n+a+b)^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1},...,\alpha_{j_r}) = O\left(\frac{1}{(n+a+b)^r}\right).$$

□

The above lemma gives the following result:

Theorem 12. *Let $A_n^{(3)}$ to be the number of α's on the third main diagonal of a random weighted staircase tableau. Then, as $n \to \infty$,*

$$A_n^{(3)} \xrightarrow{d} Pois\left(\frac{1}{2}\right).$$

Proof. Write

$$\sum_{1 \le j_1 < \ldots < j_r \le n-2} \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = \sum_{J} \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}) + \sum_{J^c} \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}),$$

where

$$J = \{1 \le j_1 < \cdots < j_r \le n-2 : \ \forall\ l = 1, \ldots, r-1; j_{l+1} - j_l \ge 3\}$$

and

$$J^c = \{1 \le j_1 < \cdots < j_r \le n-2 : \ \exists\ l = 1, \ldots, r-1; j_{l+1} - j_l < 3\}.$$

By Lemma 11,

$$\begin{aligned}\sum_J \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}) &= \sum_J \left(\prod_{l=1}^{r} \frac{b + j_{r-l+1} - 2(r-l) - 1}{(n+a+b-2(r-l)-1)_2} + O\left(\frac{1}{(n+a+b)^{r+1}}\right)\right) \\ &\approx \sum_J \prod_{l=1}^{r} \frac{j_l}{n^2} + \binom{n-2}{r} \cdot O\left(\frac{1}{n^{r+1}}\right) = \sum_{J_{r,n-2}} \prod_{l=1}^{r} \frac{j_l}{n^2} - \sum_{J_{r,n-2} \setminus J} \prod_{l=1}^{r} \frac{j_l}{n^2} + O\left(\frac{1}{n}\right).\end{aligned}$$

If $(j_1, \ldots, j_r) \in J_{r,n-2} \setminus J$ then there exists an l such that $j_{l+1} - j_l = 2$ and thus this set has $O\left(\binom{n-2}{r-1}\right)$ elements. Therefore, by Lemma 5 the expression above is asymptotic to

$$\sum_{J_{r,n-2}} \prod_{l=1}^{r} \frac{j_l}{n^2} + O\left(\binom{n-2}{r-1} \cdot \frac{n^r}{n^{2r}}\right) + O\left(\frac{1}{n}\right) = \frac{1}{2^r r!} + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty.$$

Finally, by Lemma 11,

$$\sum_{J^c} \mathbb{P}(\alpha_{j_1}, \ldots, \alpha_{j_r}) = O\left(\binom{n-2}{r-1} \frac{1}{n^r}\right) = O\left(\frac{1}{n}\right).$$

Combining these expressions with Lemma 1 completes the proof. □

4.2. **The asymptotic distribution of symbols on the third main diagonal.** In this section we prove that the total number of symbols on the third main diagonal is asymptotically Poisson with parameter 1. To this end we will prove an analog of Theorem 8 and Lemma 11. Throughout this section $x_j = x_j^{(3)}$ indicates the event that the box $(n-j-1, j)$ on the third main diagonal and in the jth column is non–empty. We need a preparatory observation.

Lemma 13. *If $j_1 \geq 3$, then*

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, 0_{n-1,1}, \beta_{n,1}) = \mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, \beta_{n-1,2}).$$

Proof. Consider

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, 0_{n-1,1}, \beta_{n,1}) &= \mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, 0_{n-1,1}, \beta_{n,1}, \alpha_{n-1,2}) \\ &\quad + \mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, 0_{n-1,1}, \beta_{n,1}, \beta_{n-1,2}).\end{aligned}$$

Notice that

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, 0_{n-1,1}, \beta_{n,1}, \alpha_{n-1,2}) = \mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, \alpha_{n,1}, \beta_{n-1,2})$$

since the second column is empty above α and thus does not restrict the first column, except for box $(n-1, 2)$ which is empty. Therefore if $\alpha_{n-1,2}$ and $\beta_{n,1}$ are switched, and the first $n-2$ boxes in column one are switched with the first $n-2$ boxes in column two, the weight does not change. Therefore,

$$\begin{aligned}&\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, 0_{n-1,1}, \beta_{n,1}) \\ &\quad = \mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, \alpha_{n,1}, \beta_{n-1,2}) + \mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, \beta_{n,1}, \beta_{n-1,2}) \\ &\quad = \mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}, \beta_{n-1,2}).\end{aligned}$$

□

The following gives the asymptotic joint distribution of non–zero symbols on the third main diagonal.

Lemma 14. *Let $1 \leq j_1 < ... < j_r \leq n-2$. If*

$$j_l \leq j_{l+1} - 3, \quad \forall l = 1, 2, ..., r-1 \tag{21}$$

then

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}) = \prod_{l=1}^{r} \frac{1}{n+a+b-r+l-1} + O\left(\frac{1}{(n+a+b)^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, ..., x_{j_r}) = O\left(\frac{1}{(n+a+b)^r}\right).$$

Proof. The proof is by induction on r. When $r = 1$ then by the same argument as in the beginning of the proof of Lemma 11, we get

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_{j_1}) \neq 0) &= \frac{n+a+b-3}{(n+a+b-1)_2} + O\left(\frac{1}{(n+a+b)^2}\right) \\ &= \frac{1}{n+a+b-1} + O\left(\frac{1}{(n+a+b)^2}\right)\end{aligned}$$

Assume the statement holds for integers up to $r-1$. We may and do assume that $j_1 = 1$. We again consider two cases.

Case 1. $j_2 \geq 4$.

As in (19) we split the probability into three pieces according to whether the boxes $(n-1,1)$, $(n-2,2)$ are empty or not

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1,...,x_{j_r}) &= \mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-1-1,2},x_1,...,x_{j_r})\\ &\quad+\mathbb{P}_{n,\alpha,\beta}(\beta_{n-1,1},0_{n-2,2},x_1,...,x_{j_r})\\ &\quad+\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2},x_1,...,x_{j_r}).\end{aligned} \tag{22}$$

The main contribution is the first probability which can be broken up into two cases.

Case 1.1. $x_1=\alpha$.

This is the case considered in the proof of Lemma 11 and gives

$$\begin{aligned}&\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-2,2},\alpha_1,x_{j_2},...,x_{j_r})\\ &\quad=\mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2},...,x_{j_r-2})\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-2,2},\alpha_1).\end{aligned}$$

Case 1.2. $x_1=\beta$.

Then

$$\begin{aligned}&\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-2,2},\beta_1,x_{j_2},...,x_{j_r})\\ &\quad=\mathbb{P}_{n,\alpha,\beta}(x_{j_2},...,x_{j_r}\mid 0_{n-1,1},0_{n-2,2},\beta_1)\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-2,2},\beta_1).\end{aligned}$$

As can be seen by removing the $(n-2)$nd row and the third column, the conditional probability is equal to

$$\begin{aligned}&\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1},...,x_{j_r-1}\mid 0_{n-2,1},\beta_{n-1,1})=\mathbb{P}_{n-1,\alpha,\beta}(x_{j_2-1},...,x_{j_r-1}\mid\beta_{n-2,2})\\ &\quad=\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(x_{j_2-2},...,x_{j_r-2}\mid\beta_{n-2,1}),\qquad\text{with }\widetilde{b}=b+1,\end{aligned}$$

where the first equality above follows by writing the probability of the left as the ratio and applying Lemma 13 to both numerator and denominator and the second follows from (4) by observing that at this point no entries of the first column are involved so that it can be removed.

At this point $\beta_{n-2,1}$ is a corner box of staircase tableau of size $n-2$ and we can use (11) to get

$$\begin{aligned}&\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(x_{j_2-2},...,x_{j_r-2},\beta_{n-2,1})\\ &\quad=\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(x_{j_2-2},...,x_{j_r-2})-\mathbb{P}_{n-3,\alpha,\widetilde{\beta}}(x_{j_2-3},\ldots,x_{j_r-3})\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(\alpha_{n-2,1}).\end{aligned}$$

Combining the above expressions and adding Case 1.1 and Case 1.2 we obtain

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1,\ldots,x_{j_r})&=\mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2},...,x_{j_r-2})\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-2,2},\alpha_1)\\ &+\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(x_{j_2-2},...,x_{j_r-2})\frac{\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-2,2},\beta_1)}{\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(\beta_{n-2,1})}\\ &-\mathbb{P}_{n-3,\alpha,\widetilde{\beta}}(x_{j_2-3},...,x_{j_r-3})\frac{\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1},0_{n-2,2},\beta_1)\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(\alpha_{n-2,1})}{\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(\beta_{n-2,1})}.\end{aligned} \tag{23}$$

Note that by (4)

$$\mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2},...,x_{j_r-2})=\mathbb{P}_{n-3,\alpha,\widetilde{\beta}}(x_{j_2-3},...,x_{j_r-3}). \tag{24}$$

Now, using (4) with $\widehat{a} = n + a - 3$, we calculate directly,

$$\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1}, 0_{n-2,2}, \alpha_1) = \mathbb{P}_{3,\widehat{a},\beta}(0_{2,1}, 0_{1,2}, \alpha_1) = \frac{b}{(n+a+b-1)_2} \tag{25}$$

$$\mathbb{P}_{n,\alpha,\beta}(0_{n-1,1}, 0_{n-2,2}, \beta_1) = \frac{n+a-3}{(n+a+b-1)_2}. \tag{26}$$

Also, by (2)

$$\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(\alpha_{n-2,1}) = \frac{\widetilde{b}}{n-2+a+\widetilde{b}-1} = \frac{b+1}{n+a+b-2}, \tag{27}$$

$$\mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(\beta_{n-2,1}) = \frac{n+a-3}{n+a+b-2}. \tag{28}$$

So by substituting Equations (24)-(28) into Equation (23) and simplifying

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1, \ldots, x_{j_r}) &= \mathbb{P}_{n-2,\alpha,\widetilde{\beta}}(x_{j_2-2}, ..., x_{j_r-2}) \frac{1}{n+a+b-1} \\ &+ \mathbb{P}_{n-3,\alpha,\widetilde{b}}(x_{j_2-3}, ..., x_{j_r-3}) \cdot \left(\frac{b}{(n+a+b-1)_2} - \frac{b+1}{(n+a+b-1)_2} \right).\end{aligned}$$

The second summand by the induction hypothesis is

$$O\left(\frac{1}{(n+a+b)^{r-1}} \cdot \frac{1}{(n+a+b)^2} \right) = O\left(\frac{1}{(n+a+b)^{r+1}} \right).$$

If $j_l \leq j_{l+1} - 3$ for all $l = 2, \ldots, r-1$ then by the induction hypothesis, the first summand is

$$\begin{aligned}&\left(\prod_{l=1}^{r-1} \frac{1}{n-2+a+\widetilde{b}-(r-1)+l-1} + O\left(\frac{1}{(n-2+a+\widetilde{b})^r} \right) \right) \frac{1}{n+a+b-1} \\ &= \left(\prod_{l=1}^{r-1} \frac{1}{n+a+b-r+l-1} + O\left(\frac{1}{(n+a+b-1)^r} \right) \right) \frac{1}{n+a+b-1} \\ &= \prod_{l=1}^{r} \frac{1}{n+a+b-r+l-1} + O\left(\frac{1}{(n+a+b-1)^{r+1}} \right).\end{aligned}$$

On the other hand, if for some $2 \leq l \leq r-1$, $j_l > j_{l+1} - 3$ then by the induction hypothesis again the same expression is,

$$O\left(\frac{1}{(n-2+a+\widetilde{b})^{r-1}} \right) \frac{1}{n+a+b-1} = O\left(\frac{1}{(n+a+b-1)^r} \right).$$

Now, returning to Equation (22), the first two probabilities can be calculated in a similar manner to obtain an equation like Equation (23). For example, $\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, x_1, ..., x_{j_r})$ is

$$\begin{aligned}&\mathbb{P}_{n-3,\alpha,\beta}(x_{j_2-3}, ..., x_{j_r-3}) \mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \alpha_1) \\ &\quad + \mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2}, ..., x_{j_r-2} | \beta_{n-2,1}) \mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \beta_1) \\ &= \mathbb{P}_{n-3,\alpha,\beta}(x_{j_2-3}, ..., x_{j_r-3}) \mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \alpha_1) \\ &\quad + \mathbb{P}_{n-2,\alpha,\beta}(x_{j_2-2}, ..., x_{j_r-2}) \frac{\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \beta_1)}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1})} \\ &\quad - \mathbb{P}_{n-3,\alpha,\beta}(x_{j_2-2}, ..., x_{j_r-2}) \frac{\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2}, \beta_1) \mathbb{P}_{n-2,\alpha,\beta}(\alpha_{n-2,1})}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1})}.\end{aligned}$$

By (4) and (2)

$$\frac{\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2},\beta_1)}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1})} = \frac{\frac{a+n-3}{(n+a+b-1)_3}}{\frac{n+a-3}{n+a+b-3}} = O\left(\frac{1}{(n+a+b)^2}\right)$$

and

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2},\alpha_1) - \frac{\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2},\beta_1)\mathbb{P}_{n-2,\alpha,\beta}(\alpha_{n-2,1})}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1})}\\ = \frac{b}{(n+a+b-1)_3} - \frac{\frac{a+n-3}{(n+a+b-1)_3}\cdot\frac{b}{n+a+b-3}}{\frac{n+a-3}{n+a+b-3}} = 0.\end{aligned}$$

Therefore, by the induction hypothesis,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{n-2,2},x_1,...,x_{j_r}) = O\left(\frac{1}{(n+a+b)^{r+1}}\right).$$

In the same way,

$$\mathbb{P}_{n,\alpha,\beta}(\beta_{n-1,1},0_{n-2,2},x_1,...,x_{j_r}) = O\left(\frac{1}{(n+a+b)^{r+1}}\right).$$

Case 2. $j_2 < 4$.

The case where $j_2 = 3$ is impossible by the rules of staircase tableaux and hence has probability zero. For the case where $j_2 = 2$, by the law of total probability

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(x_1,...,x_{j_r}) &= \mathbb{P}_{n,\alpha,\beta}(x_{j_3}...,x_{j_r}|\alpha_1,\alpha_2)\mathbb{P}_{n,\alpha,\beta}(\alpha_1,\alpha_2)\\ &\quad+\mathbb{P}_{n,\alpha,\beta}(x_{j_3},...,x_{j_r}|\alpha_1,\beta_2)\mathbb{P}_{n,\alpha,\beta}(\alpha_1,\beta_2)\\ &\quad+\mathbb{P}_{n,\alpha,\beta}(x_{j_3},...,x_{j_r}|\beta_1,\alpha_2)\mathbb{P}_{n,\alpha,\beta}(\beta_1,\alpha_2)\\ &\quad+\mathbb{P}_{n,\alpha,\beta}(x_{j_3},...,x_{j_r}|\beta_1,\beta_2)\mathbb{P}_{n,\alpha,\beta}(\beta_1,\beta_2).\end{aligned} \tag{29}$$

The first term is handled in Case 2 of Lemma 11 and is $O(1/(n+a+b)^r)$. The second, by removing the last three rows of a tableau and its first, third, and fourth column is

$$\begin{aligned}&\mathbb{P}_{n-3,\alpha,\beta}(x_{j_3-3},...,x_{j_r-3} \mid \beta_{n-3,1})\mathbb{P}_{n,\alpha,\beta}(\alpha_1,\beta_2)\\ &\quad\le \mathbb{P}_{n-3,\alpha,\beta}(x_{j_3-3},...,x_{j_r-3})\frac{\mathbb{P}_{n,\alpha,\beta}(\alpha_1,\beta_2)}{\mathbb{P}_{n-3,\alpha,\beta}(\beta_{n-3,1})}.\end{aligned}$$

By (4) and (2) and remembering that b may depend on n (see (6)),

$$\frac{\mathbb{P}_{n,\alpha,\beta}(\alpha_1,\beta_2)}{\mathbb{P}_{n-3,\alpha,\beta}(\beta_{n-3,1})} = \frac{\frac{(n+a-4)b}{(n+a+b-4)^{\overline{4}}}}{\frac{n+a-4}{n+a+b-4}} = O\left(\frac{b}{(n+a+b)^3}\right) = O\left(\frac{1}{(n+a+b)^2}\right)$$

so that, by the induction hypothesis, the whole term is

$$O\left(\frac{1}{(n+a+b)^{r-2}}\cdot\frac{1}{(n+a+b)^2}\right) = O\left(\frac{1}{(n+a+b)^r}\right).$$

Finally, consider the sum of the last two terms in (29). By removing the second and third column along with the $(n-1)$st and $(n-2)$nd row for the first probability

and the third and forth column and the $(n-2)$nd and $(n-3)$rd row for the other they are

$$\begin{aligned}
&\mathbb{P}_{n-2,\alpha,\beta}(x_{j_3-2},...,x_{j_r-2} \mid \beta_{n-2,1},\alpha_{n-3,2})\mathbb{P}_{n,\alpha,\beta}(\beta_1,\alpha_2)\\
&\quad+\mathbb{P}_{n-2,\alpha,\beta}(x_{j_3-2},...,x_{j_r-2} \mid \beta_{n-2,1},\beta_{n-3,2})\mathbb{P}_{n,\alpha,\beta}(\beta_1,\beta_2)\\
&\quad=\mathbb{P}_{n-2,\alpha,\beta}(x_{j_3-2},...,x_{j_r-2},\beta_{n-2,1},\alpha_{n-3,2})\frac{\mathbb{P}_{n,\alpha,\beta}(\beta_1,\alpha_2)}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1},\alpha_{n-3,2})}\\
&\quad+\mathbb{P}_{n-2,\alpha,\beta}(x_{j_3-2},...,x_{j_r-2},\beta_{n-2,1},\beta_{n-3,2})\frac{\mathbb{P}_{n,\alpha,\beta}(\beta_1,\beta_2)}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1},\beta_{n-3,2})}.
\end{aligned}$$

We now observe that

$$\frac{\mathbb{P}_{n,\alpha,\beta}(\beta_1,\alpha_2)}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1},\alpha_{n-3,2})}=\frac{\mathbb{P}_{n,\alpha,\beta}(\beta_1,\beta_2)}{\mathbb{P}_{n-2,\alpha,\beta}(\beta_{n-2,1},\beta_{n-3,2})}=\frac{1}{(n+a+b-2)^{\overline{2}}}$$

as can be seen by applying (4) to each of the four probabilities. Further, since the box $(n-3,2)$ is on the main diagonal of a staircase tableau of size $n-2$, $\beta_{n-3,2}$ and $\alpha_{n-3,2}$ are complements of each other. Therefore, the sum of the last two terms in (29) is

$$\begin{aligned}
&\mathbb{P}_{n-2,\alpha,\beta}(x_{j_3-2},...,x_{j_r-2},\beta_{n-2,1})\frac{1}{(n+a+b-1)_2}\\
&\quad\le\mathbb{P}_{n-2,\alpha,\beta}(x_{j_3-2},...,x_{j_r-2})\frac{1}{(n+a+b-1)_2}=O\left(\frac{1}{(n+a+b)^r}\right)
\end{aligned}$$

by the induction hypothesis. □

We can now give the asymptotic distribution of the number of symbols on the third main diagonal.

Theorem 15. *Let $X_n^{(3)}$ be the number of symbols on the third main diagonal, i.e. $X_n^{(3)}:=\sum_{j=1}^{n-2}I_{x_j}$. Then, as $n\to\infty$,*

$$X_n^{(3)}\xrightarrow{d}Pois\,(1)\,.$$

Proof. As in the proof of Theorem 12 we split the sum

$$\sum_{1\le j_1<\ldots<j_r\le n-2}\mathbb{P}(x_{j_1},\ldots,x_{j_r})=\sum_{J}\mathbb{P}(x_{j_1},\ldots,x_{j_r})+\sum_{J^c}\mathbb{P}(x_{j_1},\ldots,x_{j_r}). \tag{30}$$

By Lemma 14,

$$\begin{aligned}
&\sum_{J}\mathbb{P}(x_{j_1},\ldots,x_{j_r})=\sum_{J}\left(\prod_{k=1}^{r}\frac{1}{n+a+b-r+k-1}+O\left(\frac{1}{(n+a+b)^{r+1}}\right)\right)\\
&=\left(\binom{n-2}{r}+O\left(n^{r-1}\right)\right)\left(\prod_{k=1}^{r}\frac{1}{n+a+b-r+k-1}+O\left(\frac{1}{(n+a+b)^{r+1}}\right)\right)\\
&=\frac{1}{r!}+O\left(\frac{1}{n}\right),\quad\text{as } n\to\infty.
\end{aligned}$$

Finally, since cardinality of J^c is $O\left(\binom{n-2}{r-1}\right)$, by Lemma 14,

$$\sum_{J^c}\mathbb{P}(x_{j_1},\ldots,x_{j_r})=O\left(\binom{n-2}{r-1}\frac{1}{n^r}\right)=O\left(\frac{1}{n}\right).$$

Combining the last two expressions with (30) proves that

$$\mathbb{E}(X_n)_r = r! \left(\sum_{1 \le j_1 < \ldots < j_r \le n-2} \mathbb{P}(x_{j_1}, \ldots, x_{j_r}) \right) \to 1, \quad \text{as } n \to \infty.$$

□

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