On the distribution of parameters in random weighted staircase tableaux

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Abstract. In this paper, we study staircase tableaux, a combinatorial object introduced due to its connections with the asymmetric exclusion process (ASEP) and Askey-Wilson polynomials. Due to their interesting connections, staircase tableaux have been the object of study in many recent papers. More specific to this paper, the distribution of various parameters in random staircase tableaux has been studied. There have been interesting results on parameters along the main diagonal, however, no such results have appeared for other diagonals. It was conjectured that the distribution of the number of symbols along the kth diagonal is asymptotically Poisson as k and the size of the tableau tend to infinity. We partially prove this conjecture; more specifically we prove it for the second main diagonal.

Keywords: staircase tableau, asymmetric exclusion process, Poisson distribution, weak convergence

1 Introduction

In this paper, we study staircase tableaux, a combinatorial object introduced (in [10], [11]) due to connections with the asymmetric exclusion process (ASEP) and Askey-Wilson polynomials. The ASEP can be defined as a Markov chain with n sites, with at most one particle occupying each site. Particles may jump to any neighboring empty site with rate u to the right and rate q to the left. Particles may enter and exit at the first site with rates α and γ respectively. Similarly, particles may enter and exit the last site with rates δ and β. The ASEP is an interesting particle model that has been studied extensively in mathematics and physics. It has also been studied in many other fields, including computational biology [3], and biochemistry, specifically as a primitive model for protein synthesis [16]. Staircase tableaux were introduced per a connection between the steady state distribution of the ASEP and the generating function for staircase tableaux [11]. See Section 2 for a discussion of the ASEP and its connection with staircase tableaux.

In addition to interest in its own right, the ASEP has been known to have interesting connections in combinatorics and analysis. Consequently, staircase tableaux have similarly been connected to many combinatorial objects and a family of polynomials. In fact, the generating function for staircase tableaux has been used to give a formula for the moments of Askey-Wilson polynomials [11], [7]. Staircase tableaux have also inherited many interesting properties from other types of tableaux (See [1], [5], [6], [11], [8], [9], [15], [17]). We refer to [14, Table 1] for a description of some of the bijections between the various

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types of the tableaux.

Due to all these interesting connections, staircase tableaux have been the object of study in many recent papers. In [12], a probabilistic approach was developed and the distributions of various parameters were studied. In fact, it was shown in [12] that the distribution of the number of α's and β's in a staircase tableau are asymptotically normal, and distributions regarding the main diagonal were also given (see Section 4). In [14], the distribution of each box in a staircase tableau was given, and it was conjectured that the distribution of the number of symbols along the kth diagonal is asymptotically Poisson as k and the size of the tableau tend to infinity. The main result of this paper is the proof of this in a special case when $k = n - 1$. That is, we show that the distribution of the number of symbols along the second main diagonal is asymptotically Poisson with parameter 1, see Theorem 8. Similarly, we show that the number of α's (resp. β's) along the second main diagonal is asymptotically Poisson with parameter $1/2$, see Theorem 5 and Corollary 6.

2 Definitions and Notation

Staircase tableaux were first introduced in [10] and [11] as follows:

Definition 1 *A staircase tableau of size n is a Young diagram of shape (n, n-1, ..., 1) such that:*

1. *The boxes are empty or contain an α, β, γ, or δ.*
2. *All boxes in the same column and above an α or γ are empty.*
3. *All boxes in the same row and to the left of an β or δ are empty.*
4. *Every box on the diagonal contains a symbol.*

The rows and columns in a staircase tableau are numbered from 1 through n, beginning with the box in the NW-corner and continuing south and east respectively. Each box is numbered (i, j) where $i, j \in \{1, 2, ..., n\}$. Note that $i + j \leq n + 1$. We refer to the collection of boxes $(n - i + 1, i)$ such that $i = 1, 2, ..., n$ as the main diagonal, and the collection of boxes $(n - i, i)$ such that $i = 1, 2, ..., n - 1$ as the second main diagonal.

Following the conventions of [14], $\mathcal{S}_n$ is the set of all staircase tableaux of size n. For a given $S \in \mathcal{S}_n$, the number of α's, γ's, β's and δ's in S are denoted by $N_\alpha, N_\beta, N_\gamma$, and N_δ respectively. The weight of S is the product of all symbols in S:

$$wt(S) = \alpha^{N_\alpha}\beta^{N_\beta}\gamma^{N_\gamma}\delta^{N_\delta}.$$

It was known (see e.g. [4]) that the generating function $Z_n(\alpha, \beta, \gamma, \delta) := \sum_{S\in\mathcal{S}_n} wt(S)$ is equal to the product:

$$Z_n(\alpha, \beta, \gamma, \delta) = \prod_{i=0}^{n-1}(\alpha + \beta + \delta + \gamma + i(\alpha + \gamma)(\beta + \delta)). \tag{1}$$

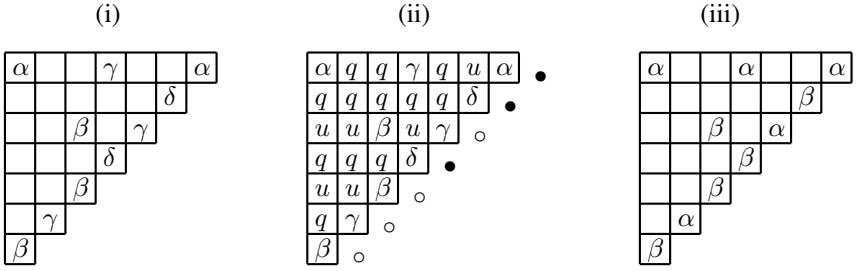

Fig. 1: A staircase tableau of size 7 with weight $\alpha^2\beta^3\delta^2\gamma^3$. (ii) The extension of (i) to a staircase tableau of weight $\alpha^2\beta^3\delta^2\gamma^3u^6q^{13}$ and type $\bullet\bullet\circ\bullet\circ\circ\circ$. (iii) The α/β-staircase tableau obtained from (i) by replacing γ's with α's and δ's with β's. It's weight is $\alpha^5\beta^5$.

Notice that there is an involution on the staircase tableaux of a given size obtained by interchanging the rows and the columns, α's and β's, and γ's and δ's, see further [4]. In particular, the fact that α's and β's are identical up to this involution allows us to extend results for α's to results for β's.

The connection between staircase tableaux and the ASEP requires an extension of this preceding definition. After following the rules from Definition 1, we then fill all the empty boxes with u's and q's, the rates at which particles in the ASEP jump to the right and left respectively. We do so by first filling all boxes to the left of a β with a u and to the left of a δ with a q. Then, we fill the empty boxes with a u if it is above an α or a δ, and q otherwise. The weight of a staircase tableau filled as such is defined in the same way, the product of the parameters in each box. Also, the total weight of all such staircase tableaux, which we denote by $\mathcal{S}_n^{'}$, is given by:

$$Z_n(\alpha,\beta,\gamma,\delta,q,u) := \sum_{S\in\mathcal{S}_n'} wt(S).$$

Then, each staircase tableau of size n is associated with a state of the ASEP with n sites (See Figure 1). This is done by aligning the Markov chain with the diagonal entries of the staircase tableau. A site is filled if the corresponding diagonal entry is an α or a γ and a site is empty if the corresponding diagonal entry is a β or a δ. Each staircase tableau's associated state of the ASEP is called its type.

Using this association, it was shown in [11] that the steady state probability that the ASEP is in state η is given by:

$$\frac{\sum_{T\in\mathfrak{T}} wt(T)}{Z_n},$$

where $\mathfrak{T}$ is the set of all staircase tableau of type η.

For the purposes of this paper, we will consider more simplified staircase tableaux, namely α/β-staircase tableaux as introduced in [14], which are staircase tableaux limited to the symbols α and β. The set $\overline{\mathcal{S}}_n \subset \mathcal{S}_n$ denotes the set of all such staircase tableaux. Since the symbols α and γ follow the same rules in the definition, as do β and δ, any $S \in \mathcal{S}_n$ can be obtained from an $S^{'} \in \overline{\mathcal{S}}_n$ by replacing the appropriate α's with γ's and β's with δ's.

The generating function of α/β-staircase tableaux is:

$$Z_n(\alpha,\beta) := \sum_{S\in\overline{\mathcal{S}}_n} wt(S) = Z_n(\alpha,\beta,0,0)$$

and it follows from (1) that it is simply:

$$Z_n(\alpha,\beta) = \alpha^n\beta^n(a+b)^{\overline{n}}$$

where $a := \alpha^{-1}$ and $b := \beta^{-1}$, a notation that will be used frequently throughout this paper, and $(a+b)^{\overline{n}}$ is the rising factorial of $(a+b)$, i.e. $(a+b)^{\overline{n}} = (a+b)((a+b)+1)\cdots((a+b)+n-1)$.

We wish to consider random staircase tableaux as was done in [12] but it suffices to study random α/β-staircase tableaux, denoted by $S_{n,\alpha,\beta}$, as was done in [14]. All of our results for random α/β-staircase tableaux can be extended to random staircase tableaux with all four parameters, $\alpha,\gamma,\beta,\delta$. This is done by considering $S_{n,\alpha+\gamma,\beta+\delta}$ and randomly replacing each α with γ with probability $\frac{\gamma}{\alpha+\gamma}$ and similarly, each β with δ with probability $\frac{\delta}{\beta+\delta}$ independently for each occurrence. Notice that $Z_n(\alpha,\beta,\gamma,\delta) = Z_n(\alpha+\gamma,\beta+\delta)$. We also allow all parameters to be arbitrary positive real numbers, i.e. $\alpha,\beta \in (0,\infty)$, allowing $\alpha=\infty$ by fixing β and taking the limit or vice versa, or $\alpha=\beta=\infty$ by taking the limit. The following is a formal definition as in [14]:

Definition 2 *For all $n \geq 1$, $\alpha,\beta \in [0,\infty)$ with $(\alpha,\beta) \neq (0,0)$, $S_{n,\alpha,\beta}$ is defined to be a random α/β-staircase tableau in $\overline{\mathcal{S}}_n$ with respect to the probability distribution on $\overline{\mathcal{S}}_n$ given by:*

$$\forall S \in \overline{\mathcal{S}}_n, \qquad \mathbb{P}(S_{n,\alpha,\beta} = S) = \frac{wt(S)}{Z_n(\alpha,\beta)} = \frac{\alpha^{N_\alpha}\beta^{N_\beta}}{Z_n(\alpha,\beta)}.$$

We will also write is as

$$\mathcal{L}(S_{n,\alpha,\beta}) = \frac{\alpha^{N_\alpha}\beta^{N_\beta}}{Z_n(\alpha,\beta)}.$$

Using this definition, Hitczenko and Janson presented the distribution of a given box in a random staircase tableau. If a box is on the main diagonal, the distribution is (see [14, Theorem 7.1]):

$$\mathbb{P}(S_{n,\alpha,\beta}(i,n+1-i)=\alpha) = \frac{n-i+b}{n+a+b-1}. \tag{2}$$

Since a box on the main diagonal is never empty, the β case follows trivially.

If a box is not on the main diagonal, its distribution is (see [14, Theorem 7.2]):

$$\mathbb{P}(S_{n,\alpha,\beta}(i,j)=\alpha) = \frac{j-1+b}{(i+j+a+b-1)(i+j+a+b-2)} \tag{3}$$

$$\mathbb{P}(S_{n,\alpha,\beta}(i,j)=\beta) = \frac{i-1+a}{(i+j+a+b-1)(i+j+a+b-2)}. \tag{4}$$

3 Subtableaux and Preliminaries

For an arbitrary $S \in \overline{\mathcal{S}}_n$ and an arbitrary box (i,j) in S, define $S[i,j]$ to be the subtableau in $\overline{\mathcal{S}}_{n-i-j+2}$ obtained by deleting the first $i-1$ rows and $j-1$ columns, see [14]. The following statement was proven in [14, Theorem 6.1] and is a useful tool in our results:

$$S_{n,\alpha,\beta}[i,j] \stackrel{d}{=} S_{n-i-j+2,\hat{a},\hat{b}}, \text{ with } \hat{a}=a+i-1 \text{ and } \hat{b}=b+j-1. \tag{5}$$

The following two lemmas consider the probability of an arbitrary staircase tableau in $\overline{\mathcal{S}}_n$ that is conditioned on having an α or a β in the box $(n-1,1)$. The statements follow almost immediately from the definition of a staircase tableau, but will be used frequently throughout the paper.

Lemma 1 *If $S_{n,\alpha,\beta}$ is conditioned on $S_{n,\alpha,\beta}(n-1,1)=\alpha$, then the subtableau $S_{n,\alpha,\beta}[1,3] \stackrel{d}{=} S_{n-2,\alpha,\beta}$, that is*

$$\mathcal{L}(S_{n,\alpha,\beta} \mid S_{n,\alpha,\beta}(n-1,1)=\alpha) = \mathcal{L}(S_{n-2,\alpha,\beta}).$$

Proof: If $S_{n,\alpha,\beta}$ is a staircase tableau such that the box $S_{n,\alpha,\beta}(n-1,1)=\alpha$, then the box, $S_{n,\alpha,\beta}(n,1)=\beta$ and $S_{n,\alpha,\beta}(n-1,2)=\alpha$ by the rules of a staircase tableau. The first and second column are otherwise empty by those same rules. The remainder, $S_{n,\alpha,\beta}[1,3]$, is an arbitrary staircase tableau of size $n-2$. Therefore, the lemma follows. □

Lemma 2 *Let $(S_{n,\alpha,\beta})_{i,j}$ be a tableau $S_{n,\alpha,\beta}$ with the ith row and the jth column removed. If $S_{n,\alpha,\beta}$ is conditioned on $S_{n,\alpha,\beta}(n-1,1)=\beta$, then the subtableau $(S_{n,\alpha,\beta})_{n-1,2} \stackrel{d}{=} \widetilde{S}_{n-1,\alpha,\beta}$ where $\widetilde{S}_{n-1,\alpha,\beta}$ is random tableau of size $n-1$ conditioned on having a β in the $(n-1,1)$ box. In other words*

$$\mathcal{L}(S_{n,\alpha,\beta} \mid S_{n,\alpha,\beta}(n-1,1)=\beta) = \mathcal{L}(S_{n-1,\alpha,\beta} \mid S_{n-1,\alpha,\beta}(n-1,1)=\beta).$$

Proof: If $S_{n,\alpha,\beta}$ is a staircase tableau such that the box $S_{n,\alpha,\beta}(n-1,1)=\beta$, then the box $S_{n,\alpha,\beta}(n-1,2)=\alpha$ and $S_{n,\alpha,\beta}(n-1,1)=\beta$ by the rules of a staircase tableau. The second column is otherwise empty by those same rules. The nth row only has one box, $(n,1)$, which must be a β. The remainder is an arbitrary staircase tableau of size $n-1$ conditioned to have a β in box $(n-1,1)$. Therefore, the lemma follows. □

4 Distribution of parameters along the second main diagonal

The asymptotic distribution of parameters along the main diagonal is known. The number of α/γ symbols and the number of β/δ symbols along the main diagonal were proven to be asymptotically normal in [12], and the distribution of boxes along the main diagonal was given in [14]. However, the distributions of parameters on the other diagonals have not been studied specifically. The expected values were computed in [14] and it was conjectured there that the asymptotic distribution for the symbols on the kth diagonal is Poisson as $k=k_n$ goes to infinity with n going to infinity. As the first step towards proving that conjecture we now present results concerning the second main diagonal. In order to simplify notation, let $S_{n,\alpha,\beta}(i)$ be the symbol contained in second main diagonal box $(n-i,i)$ of $S_{n,\alpha,\beta}$. As our first result, the following is the distribution of boxes along the second main diagonal.

Theorem 3 *Let $1 \le j_1 < ... < j_r \le n-1$. If*

$$j_k \le j_{k+1}-2, \quad \forall k = 1,2,...,r-1 \tag{6}$$

then

$$\mathbb{P}(S_{n,\alpha,\beta}(j_1) = ... = S_{n,\alpha,\beta}(j_r) = \alpha) = \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(a+b+n-2r+2k-1)(a+b+n-2r+2k-2)}.$$

(For $r=1$, this is (3)). Otherwise,

$$\mathbb{P}(S_{n,\alpha,\beta}(j_1) = ... = S_{n,\alpha,\beta}(j_r) = \alpha) = 0.$$

Proof: First note that when (6) fails there exists j_k such that $j_k = j_{k+1} - 1$ and thus there must be two α's in boxes side by side on the $(n-i, i)$ diagonal. But this is impossible by the rules of a staircase tableau as no symbol can be put in the diagonal box $(n - j_k, j_{k+1})$ adjacent to these two boxes. Therefore the probability is 0.

Suppose now that (6) holds. We proceed by induction on r. By (5), $S_{n,\alpha,\beta}[1, j_1] \stackrel{d}{=} S_{n-j_1+1,\alpha,\hat{\beta}}$ with $\hat{\beta}^{-1} = \beta^{-1} + j_1 - 1$ which yields:

$$\begin{aligned}\mathbb{P}(S_{n,\alpha,\hat{\beta}}(j_1) = ... = S_{n,\alpha,\hat{\beta}}(j_r) = \alpha) &= \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) = ... = S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) = \alpha) \\ &= \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(j_2 - j_1 + 1) = ... = S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) = \alpha \mid S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha) \\ &\quad \cdot \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha).\end{aligned}$$

By Lemma 1 and the induction hypothesis:

$$\begin{aligned}&\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(j_2 - j_1 + 1) = ... = S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) = \alpha \mid S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha) \\ &\quad = \mathbb{P}(S_{n-j_1-1,\alpha,\hat{\beta}}(j_2 - j_1 - 1) = ... = S_{n-j_1-1,\alpha,\hat{\beta}}(j_r - j_1 - 1) = \alpha) \\ &\quad = \prod_{k=1}^{r-1} \frac{\hat{b} + j_{r-k+1} - j_1 - 2r + 2k}{(a + \hat{b} + n - j_1 - 2r + 2k)(a + \hat{b} + n - j_1 - 2r + 2k - 1)}.\end{aligned}$$

By (3):

$$\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha) = \frac{\hat{b}}{(n - j_1 + a + \hat{b})(n - j_1 + a + \hat{b} - 1)}.$$

Therefore,

$$\begin{aligned}&\mathbb{P}(S_{n,\alpha,\hat{\beta}}(j_1) = ... = S_{n,\alpha,\hat{\beta}}(j_r) = \alpha) \\ &\quad = \prod_{k=1}^{r} \frac{\hat{b} + j_{r-k+1} - j_1 - 2r + 2k}{(a + \hat{b} + n - j_1 - 2r + 2k)(a + \hat{b} + n - j_1 - 2r + 2k - 1)} \\ &\quad = \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(a + b + n - 2r + 2k - 1)(a + b + n - 2r + 2k - 2)}\end{aligned}$$

which proves the result. □

Our second main result is the distribution of the number of α's (and β's) along the second main diagonal. The proof requires a lemma.

Lemma 4 *Let*

$$J_{r,m} := \{1 \le j_1 < ... < j_r \le m : \ j_k \le j_{k+1} - 2, \ \ \forall k = 1, 2, ..., r-1\}.$$

Then

$$\sum_{J_{r,m}} \left(\prod_{k=1}^{r} j_{r-k+1} \right) = \frac{(m+1)_{2r}}{2^r r!},$$

where $(x)_r = x(x-1)\dots(x-(r-1))$ *is the falling factorial.*

Proof: By induction on r. When $r = 1$:

$$\sum_{J_{1,m}} \left(\prod_{k=1}^{1} j_{1-k+1} \right) = \sum_{j_1=1}^{m} j_1 = \frac{(m+1)m}{2}.$$

Assume the statement holds for $r-1$. Then:

$$\begin{aligned} \sum_{J_{r,m}} \left(\prod_{k=1}^{r} j_{r-k+1} \right) &= \sum_{j_r=2r-1}^{m} j_r \left(\sum_{J_{r-1,j_r-2}} \prod_{k=2}^{r} j_{r-k+1} \right) = \sum_{j_r=2r-1}^{m} j_r \frac{(j_r-1)_{2(r-1)}}{2^{r-1}(r-1)!} \\ &= \frac{1}{2^{r-1}(r-1)!} \sum_{j_r=2r-1}^{m} (j_r)_{2r-1} \end{aligned}$$

where the second equality is by the induction hypothesis. Since

$$\sum_{j_r=2r-1}^{m} (j_r)_{2r-1} = \sum_{j_r=0}^{m} (j_r)_{2r-1}$$

the lemma will be proved once we verify that

$$\sum_{j=0}^{m} (j)_t = \frac{(m+1)_{t+1}}{t+1},$$

for any non-negative integer t (and apply it with $t = 2r-1$). Using the identity

$$\sum_{j=0}^{m} \binom{j}{t} = \binom{m+1}{t+1}$$

(see, e.g. [13, Formula (5.10)]) we see that

$$\sum_{j=0}^{m} (j)_t = \sum_{j=0}^{m} \frac{j!}{(j-t)!} = t! \sum_{j=0}^{m} \binom{j}{t} = t! \binom{m+1}{t+1} = t! \frac{(m+1)_{t+1}}{(t+1)!} = \frac{(m+1)_{t+1}}{m+1},$$

as asserted. □

Finally, define A_n and B_n to be the number of α's and β's on the second main diagonal, i.e. $A_n := \sum_{j=1}^{n-1} I_{S_{n,\alpha,\beta}(j)=\alpha}$ and $B_n := \sum_{j=1}^{n-1} I_{S_{n,\alpha,\beta}(j)=\beta}$. Then, the asymptotic distribution of A_n and B_n is given in the following theorem and corollary.

Theorem 5 *Let $Pois(\lambda)$ be a Poisson random variable with parameter λ. Then, as $n \to \infty$,*

$$A_n \xrightarrow{d} Pois\left(\frac{1}{2}\right). \tag{7}$$

Proof: By [2, Theorem 20, Chapter 1] it suffices to show that the rth factorial moment of A_n satisfies:

$$\mathbb{E}(A_n)_r \to \left(\frac{1}{2}\right)^r \quad \text{as } \to \infty. \tag{8}$$

For the ease of notation let $I_j := I_{S_{n,\alpha,\beta}(j)=\alpha}$ and consider

$$\begin{aligned} z^{A_n} &= z^{\sum_{j=1}^{n-1} I_j} = \prod_{j=1}^{n-1} z^{I_j} = \prod_{j=1}^{n-1} (1+(z-1))^{I_j} = \prod_{j=1}^{n-1} (1+I_j(z-1)) \\ &= 1 + \sum_{r=1}^{n-1} \left(\sum_{1\le j_1<\ldots<j_r\le n-1} \left(\prod_{k=1}^{r} I_{j_k} \right) \right) (z-1)^r \\ &= 1 + \sum_{r=1}^{n-1} (z-1)^r \left(\sum_{1\le j_1<\ldots<j_r\le n-1} \left(\prod_{k=1}^{r} I_{j_k} \right) \right). \end{aligned}$$

Thus,

$$\mathbb{E}(z^{A_n}) = 1 + \sum_{r=1}^{n-1} (z-1)^r \left(\sum_{1\le j_1<\ldots<j_r\le n-1} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}) \right).$$

Hence

$$\mathbb{E}(A_n)_r = \frac{d^r}{dz^r}(\mathbb{E}z^{A_n})|_{z=1} = r! \left(\sum_{1\le j_1<\ldots<j_r\le n-1} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}) \right).$$

By Theorem 3 and Lemma 4

$$\begin{aligned} \mathbb{E}(A_n)_r &= r! \sum_{J_{r,n-1}} \left(\prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(a+b+n-2r+2k-1)(a+b+n-2r+2k-2)} \right) \\ &\approx r! \sum_{J_{r,n-1}} \left(\prod_{k=1}^{r} \frac{j_{r-k+1}}{n^2} \right) = \frac{r!}{n^{2r}} \frac{(n)_{2r}}{2^r r!} \to \left(\frac{1}{2}\right)^r, \quad \text{as } n \to \infty. \end{aligned}$$

□

Corollary 6 *The rth factorial moment of the number B_n of β's on the second main diagonal of a random staircase tableau of size n satisfies:*

$$\mathbb{E}(B_n)_r \to \left(\frac{1}{2}\right)^r \quad \textit{as } n \to \infty. \tag{9}$$

Furthermore,

$$B_n \xrightarrow{d} Pois\left(\frac{1}{2}\right) \quad \textit{as } n \to \infty. \tag{10}$$

Proof: This follows by symmetry, see Section 1. □

Remark 1 *Theorem 5 and Corollary 6 hold regardless of the values of α and β including the cases discussed earlier when $\alpha = \infty$, $\beta = \infty$, or $\alpha = \beta = \infty$. As noted in [14, Examples 3.6 and 3.7] these cases correspond to staircase tableaux with the maximal number of α's (or β's) and the maximal number of symbols, respectively.*

5 Distribution of Non-Empty Boxes

The number of α's and the number of β's, N_a and N_b, are not independent random variables, and the second main diagonal may have empty boxes. Therefore, in order to completely describe the second main diagonal, we must consider both symbols collectively. First, we present the distribution of non-empty boxes along the second main diagonal.

Theorem 7 *Let $1 \leq j_1 < ... < j_r \leq n-1$. If (6) holds then*

$$\mathbb{P}(S_{n,\alpha,\beta}(j_1) \neq 0, ..., S_{n,\alpha,\beta}(j_r) \neq 0) = \prod_{k=1}^{r} \frac{1}{(n+a+b-r+k-1)}.$$

(For $r = 1$, this is obtained by adding (3) and (4)). Otherwise,

$$\mathbb{P}(S_{n,\alpha,\beta}(j_1) \neq 0, ..., S_{n,\alpha,\beta}(j_r) \neq 0) = 0.$$

Proof: Suppose (6) holds. We proceed by induction on r.

By (5), $S_{n,\alpha,\beta}[1, j_1] \stackrel{d}{=} S_{n-j_1+1,\alpha,\hat{\beta}}$ with $\hat{\beta}^{-1} = \beta^{-1} + j_1 - 1$ which yields:

$$\begin{aligned}&\mathbb{P}(S_{n,\alpha,\hat{\beta}}(j_1) \neq 0, ..., S_{n,\alpha,\beta}(j_r) \neq 0)\\ &\quad = \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) \neq 0, ..., S_{n-j_r+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) \neq 0).\end{aligned}$$

Further

$$\begin{aligned}&\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) \neq 0, ..., S_{n-j_r+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) \neq 0)\\ &= \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha, S_{n-j_1+1,\alpha,\hat{\beta}}(j_2 - j_1 + 1) \neq 0, ..., S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) \neq 0)\\ &\quad + \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \beta, S_{n-j_1+1,\alpha,\hat{\beta}}(j_2 - j_1 + 1) \neq 0, ..., S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) \neq 0)\\ &= \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(j_2 - j_1 + 1) \neq 0, ..., S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) \neq 0) \mid S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha)\\ &\quad \cdot \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha)\\ &\quad + \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(j_2 - j_1 + 1) \neq 0, ..., S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) \neq 0 \mid S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \beta)\\ &\quad \cdot \mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \beta).\end{aligned}$$

Now consider two cases:

Case 1: $S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha$. By Lemma 1 and the induction hypothesis:

$$\begin{aligned}&\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(j_2 - j_1 + 1) \neq 0, ..., S_{n-j_1+1,\alpha,\hat{\beta}}(j_r - j_1 + 1) \neq 0) \mid S_{n-j_1+1,\alpha,\hat{\beta}}(1) = \alpha)\\ &\quad = \mathbb{P}(S_{n-j_1-1,\alpha,\hat{\beta}}(j_2 - j_1 - 1) \neq 0, ..., S_{n-j_1-1,\alpha,\hat{\beta}}(j_r - j_1 - 1) \neq 0)\\ &\quad = \prod_{k=1}^{r-1} \frac{1}{(n-j_1+a+\hat{b}-r+k-1)}.\end{aligned}$$

Also, by (3),

$$\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1)=\alpha)=\frac{\hat{b}}{(n-j_1+a+\hat{b})(n-j_1+a+\hat{b}-1)}.$$

Therefore,

$$\begin{aligned}&\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1)=\alpha, S_{n-j_1+1,\alpha,\hat{\beta}}(j_2-j_1+1)\neq 0,...,S_{n-j_r+1,\alpha,\hat{\beta}}(j_r-j_1+1)\neq 0)\\ &=\frac{\hat{b}}{(n-j_1+a+\hat{b})(n-j_1+a+\hat{b}-1)}\cdot\prod_{k=1}^{r-1}\frac{1}{(n-j_1+a+\hat{b}-r+k-1)}.\end{aligned}$$

Case 2: $S_{n-j_1+1,\alpha,\hat{\beta}}(1)=\beta$. By Lemma 2

$$\begin{aligned}&\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(j_2-j_1+1)\neq 0,...,S_{n-j_1+1,\alpha,\hat{\beta}}(j_r-j_1+1)\neq 0 \mid S_{n-j_1+1,\alpha,\hat{\beta}}(1)=\beta)\\ &=\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(j_2-j_1)\neq 0,...,S_{n-j_{1r},\alpha,\hat{\beta}}(j_r-j_1)\neq 0 \mid S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\beta)\\ &=\frac{\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(j_2-j_1)\neq 0,...,S_{n-j_1,\alpha,\hat{\beta}}(j_r-j_1)\neq 0,\ S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\beta)}{\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\beta)}.\end{aligned}$$

The numerator is equal to

$$\begin{aligned}&\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(j_2-j_1)\neq 0,...,S_{n-j_1,\alpha,\hat{\beta}}(j_r-j_1)\neq 0)\\ &\quad-\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(j_2-j_1)\neq 0,...,S_{n-j_1,\alpha,\hat{\beta}}(j_r-j_1)\neq 0,\ S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\alpha)\\ &\quad=\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(j_2-j_1)\neq 0,...,S_{n-j_1,\alpha,\hat{\beta}}(j_r-j_1)\neq 0)\\ &\quad-\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(j_2-j_1)\neq 0,...,S_{n-j_1,\alpha,\hat{\beta}}(j_r-j_1)\neq 0 \mid S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\alpha)\\ &\qquad\cdot\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\alpha).\end{aligned}$$

By [14, Lemma 7.5] and the induction hypothesis the conditional probability above is

$$\mathbb{P}(S_{n-j_1-1,\alpha,\hat{\beta}}(j_2-j_1-1)\neq 0,...,S_{n-j_1-1,\alpha,\hat{\beta}}(j_r-j_1-1)\neq 0)=\prod_{k=1}^{r-1}\frac{1}{n-j_1+a+\hat{b}-r+k-1}. \tag{11}$$

By (4), (2) (and some algebra), the induction hypothesis, and (3), respectively,

$$\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(1)=\beta)=\frac{n-j_1+a-1}{(n-j_1+a+\hat{b})(n-j_1+a+\hat{b}-1)} \tag{12}$$

$$\frac{1}{\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\beta)}=\frac{n-j_1+a+\hat{b}-1}{n-j_1+a-1} \tag{13}$$

$$\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(j_2-j_1)\neq 0,...,S_{n-j_1+1,\alpha,\hat{\beta}}(j_r-j_1)\neq 0)=\prod_{k=1}^{r-1}\frac{1}{n-j_1+a+\hat{b}-r+k} \tag{14}$$

$$\mathbb{P}(S_{n-j_1,\alpha,\hat{\beta}}(n-j_1,1)=\alpha)=\frac{\hat{b}}{(n-j_1+a+\hat{b}-1)}. \tag{15}$$

Combining (11) - (15):

$$\begin{aligned}&\mathbb{P}(S_{n-j_1+1,\alpha,\hat{\beta}}(j_2-j_1+1)\neq 0,...,S_{n-j_1+1,\alpha,\hat{\beta}}(j_r-j_1+1)\neq 0, S_{n-j_1+1,\alpha,\hat{\beta}}(1)=\beta)\\ &= \frac{1}{n-j_1+a+\hat{b}}\cdot\Big(\prod_{k=1}^{r-1}\frac{1}{n-j_1+a+\hat{b}-r+k}\\ &\quad -\frac{\hat{b}}{n-j_1+a+\hat{b}-1}\prod_{k=1}^{r-1}\frac{1}{n-j_1+a+\hat{b}-r+k-1}\Big).\end{aligned}$$

Adding Case 1 and Case 2:

$$\begin{aligned}\mathbb{P}(S_{n,\alpha,\hat{\beta}}(j_1)\neq 0,...,S_{n,\alpha,\hat{\beta}}(j_r)\neq 0) &= \frac{1}{n-j_1+a+\hat{b}}\cdot\prod_{k=1}^{r-1}\frac{1}{n-j_1+a+\hat{b}-r+k}\\ &= \prod_{k=1}^{r}\frac{1}{n-j_1+a+\hat{b}-r+k}=\prod_{k=1}^{r}\frac{1}{n+a+b-r+k-1}\end{aligned}$$

which proves our assertion when (6) holds.

If there exists j_k such that $j_k = j_{k+1}-1$, then $\{S_{n,\alpha,\beta}(j_1)\neq 0,...,S_{n,\alpha,\beta}(j_r)\neq 0\}$ implies that two boxes side by side on the $(n-i,i)$ diagonal are non-empty, which is impossible by the rules of a staircase tableau. Therefore the probability is 0. □

As our final result, we consider the number of symbols on the second main diagonal, which we denote by X_n. Then $X_n = \sum_{j=1}^{n-1} I_j$ where $I_j := I_{S_{n,\alpha,\beta}(j)\neq 0}$. The asymptotic distribution of the number of symbols on the second main diagonal is given by:

Theorem 8 *As $n\to\infty$,*

$$X_n \xrightarrow{d} Pois\,(1)\,. \tag{16}$$

Proof: By Theorem 7 and the same argument as in Theorem 5

$$\begin{aligned}\mathbb{E}(X_n)_r &= r!\sum_{1\leq j_1<\ldots<j_r\leq n-1}\mathbb{P}(I_{j_1}\cap\ldots\cap I_{j_r}) = r!|J_{r,n-1}|\prod_{k=1}^{r}\frac{1}{n+a+b-r+k-1}\\ &= r!\left(\binom{n-1}{r}+O(n^{r-1})\right)\prod_{k=1}^{r}\frac{1}{n+a+b-r+k-1}\approx r!\cdot\frac{(n-1)^r}{r!n^r}\to 1\quad\text{as } n\to\infty.\end{aligned}$$

The result then follows by [2, Theorem 20], as discussed in the proof of Theorem 5. □

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