THE ASYMPTOTIC DISTRIBUTION OF SYMBOLS ON DIAGONALS OF RANDOM WEIGHTED STAIRCASE TABLEAUX

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Abstract. Staircase tableaux are combinatorial objects that were first introduced due to a connection with the asymmetric simple exclusion process (ASEP) and Askey-Wilson polynomials. Since their introduction, staircase tableaux have been the object of study in many recent papers. Relevant to this paper, Hitczenko and Janson proved that distribution of symbols on the first diagonal is asymptotically normal. In addition, they conjectured that other diagonals would be asymptotically Poisson. Since then, only the second and the third diagonal were proven to follow the conjecture. This paper builds upon those results to prove the conjecture for the kth diagonal where k is fixed. In particular, we prove that the distribution of the number of α's (β's) on the kth diagonal, $k > 1$, is asymptotically Poisson with parameter $1/2$. In addition, we prove that symbols on the kth diagonal are asymptotically independent and thus, collectively follow the Poisson distribution with parameter 1.

1. Introduction

This paper is on staircase tableaux, combinatorial objects which were introduced in [10] and [11]. Staircase tableaux were initially introduced to provide a combinatorial formula for the stationary distribution of the asymmetric simple exclusion process (ASEP), (see [10, Theorem 2] and [11, Theorem 3.5]). The ASEP is an important model from statistical mechanics that has been used to model protein synthesis [20], traffic flow [21], and much more (See, for example [3], [13]). We refer to [11, Section 2-3] for more details on the ASEP and its connection to staircase tableaux.

Staircase tableaux also have interesting connections in combinatorics and analysis. In the original papers ([10] and [11]), staircase tableaux were used to compute the moments of Askey-Wilson polynomials ([10, Theorem 3]), an important family of orthogonal polynomials. In addition, staircase tableaux are in bijection with permutation tableaux, alternative tableaux, and tree-like tableaux (see [15, Table 1] for more details).

These connections have been the motivation for many recent papers to focus specifically on staircase tableaux ([4], [7], [12], [15], [17], and [18]). Relevant to this paper, the distribution of symbols along the first diagonal of staircase tableaux was proven to be asymptotically normal in [15]. In addition, the expected number of symbols on any diagonal was computed which prompted a conjecture that the distribution of the kth diagonal is asymptotically Poisson as $n, k \to \infty$.

This conjecture was partially proven in [17]. More specifically, symbols on the second and third diagonal were proven to be asymptotically Poisson. The computations in [17] became increasingly difficult from just the second diagonal to the third diagonal, and consequently, the rest of the conjecture could not be proven until a different approach was discovered. This paper builds upon those results to prove the conjecture for fixed k by introducing a systematic way to calculate the weights of tableaux being considered. In particular, this approach allows us to prove that the distribution of the number of α's (β's) on the kth diagonal is asymptotically Poisson with parameter $1/2$ (Theorem 8 (Corollary 9)). In addition, we prove that symbols on the kth diagonal are asymptotically independent and thus, collectively follow the Poisson distribution with parameter 1 (Theorem 14).

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2. Definitions, Notation, and Previous Results

Definition 1. *Staircase tableaux of size n are defined to be a Young diagram of shape $n, n-1, \ldots, 1$ whose boxes are either empty or filled with an α, β, γ or δ. The filling rules are as follows:*

Key words and phrases. Staircase tableaux, asymmetric simple exclusion process, Poisson distribution.

- *Each box along the first diagonal must contain a symbol.*
- *All boxes in the same column and north of an α or γ must be empty.*
- *All boxes in the same row and west of a β or δ must be empty.*

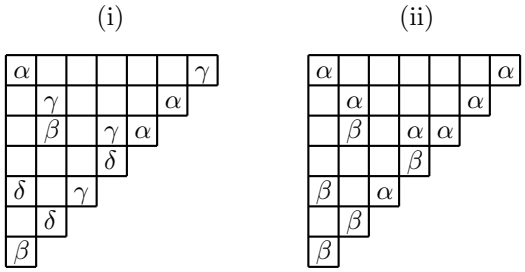

FIGURE 1. A staircase tableau of size 7 with weight $\alpha^3\beta^2\delta^3\gamma^4$. (ii) The α/β-staircase tableau version of (i) obtained by replacing γ's with α's and δ's with β's.

We enumerate the rows and columns of staircase tableaux beginning in the NW-corner. Let (i, j) refer to the box in the ith row and the jth column. We are especially interested in the diagonals of staircase tableaux and more formally define the kth diagonal to be the collection of boxes (i, j) such that $i + j = n - k + 2$. It is important to note that the kth diagonal in this paper is the $(n - k + 1)$th diagonal in [15]. We find it convenient to enumerate diagonal boxes by their column position i.e. we write $\{(n - k - l + 2, l) : \ 1 \le l \le n - k + 1\}$ for the boxes on the kth diagonal. Throughout the paper, x_j^k will denote the event (or its indicator) that there is a symbol x on the kth diagonal in the jth column of a staircase tableau; that is, there is an x in the $(n - k - j_1 + 2, j)$th box of the tableau.

Without loss of generality, we can replace all γ's with α's and δ's with β's to obtain α/β-staircase tableaux. Let S_n denote the set of all α/β-staircase tableaux. The weight generating function for staircase tableaux is well-known and the following version for S_n is from [15],

$$Z_n(\alpha, \beta) := \sum_{S \in \mathcal{S}_n} wt(S) = \alpha^n \beta^n (a + b + n - 1)_n$$

where $wt(S)$ is the product of all symbols in S, $a := \alpha^{-1}$, $b := \beta^{-1}$ and $(x)_n = x(x - 1) \dots (x - (n - 1))$ is the falling factorial of x.

It is important to note that there is a simple involution between staircase tableaux of a particular size. If we interchange the rows and columns and α's and β's, we obtain a staircase tableau of the same size. We will use this involution to obtain the distribution of β's from the distribution of α's.

We want to consider random weighted α/β-staircase tableaux as was done in [15],

Definition 2. *For all $n \ge 1$, $\alpha, \beta \in [0, \infty)$ with $(\alpha, \beta) \ne (0, 0)$, define a random $S \in \mathcal{S}_n$ to be the α/β-staircase tableau chosen according to the following distribution,*

$$\mathbb{P}_{n,\alpha,\beta}(S) := \frac{wt(S)}{Z_n(\alpha, \beta)}, \quad S \in \mathcal{S}.$$

We allow $\alpha \to \infty$ ($\beta \to \infty$) by fixing β (α) and taking the limit, or we also allow $\alpha = \beta \to \infty$. Even though we are using a probability measure on S_n, all of our results can be extended to tableaux as defined in Definition 1, i.e. staircase tableaux with all four symbols. This can be done by randomly replacing each α with γ with probability $\frac{\gamma}{\alpha+\gamma}$ and each β with δ with probability $\frac{\delta}{\beta+\delta}$ independently for each occurrence.

Using the probability measure from Definition 2, the distribution of boxes was given in [15]. For boxes on the first diagonal ([15, Theorem 7.1]),

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_j^1) = \frac{k + j + b - 2}{n + a + b - 1}, \quad \mathbb{P}_{n,\alpha,\beta}(\beta_j^1) = \frac{n - k - j + a + 1}{n + a + b - 1}.$$

For boxes on the kth diagonal, $k > 1$ ([15, Theorem 7.2]),

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_j^k) = \frac{b + j - 1}{(n - k + a + b + 1)_2}, \qquad \mathbb{P}_{n,\alpha,\beta}(\beta_j^k) = \frac{n - k - j + a + 1}{(n - k + a + b + 1)_2}. \tag{1}$$

Of course boxes on the kth diagonal may be empty if $k > 1$, and this probability is the remainder of the above.

One more useful result from [15] relates to subtableaux. For arbitrary $S \in \mathcal{S}_n$, let $S[i,j]$ be the subtableau of S that begins in box (i,j), i.e. the first $i-1$ rows and $j-1$ columns of S are deleted. Then, as proven in [15, Theorem 6.1], if $\widetilde{S} \in \mathcal{S}_{n-i-j+2}$ then

$$\mathbb{P}_{n,\alpha,\beta}(S \in \mathcal{S}_n : \ S[i,j] = \widetilde{S}) = \mathbb{P}_{n-i-j+2,\widehat{\alpha},\widehat{\beta}}(\widetilde{S}), \tag{2}$$

where $\widehat{\alpha}^{-1} = \alpha^{-1} + i - 1$ and $\widehat{\beta}^{-1} = \beta^{-1} + j - 1$. For convenience, we will often use $a := \alpha^{-1}$ and $b := \beta^{-1}$.

The distributions of the second and third diagonals were shown to converge to a Poisson random variable in [17] by the method of factorial moments. We will also use this method for the kth diagonal in general, and the following two lemmas will be useful. The first is a classical result for random graphs (see for example, [19]) and the second is the multivariate analogue.

Lemma 1. *Let $Y = \sum_{j=1}^{m} I_j$, where (I_j) are indicator random variables. Then, for $r \geq 1$,*

$$\mathbb{E}(Y)_r = r! \sum_{1 \leq j_1 < \ldots < j_r \leq m} \mathbb{P}(I_{j_1} \cap \ldots \cap I_{j_r}),$$

where $\mathbb{E}(X)_r = \mathbb{E}[X(X-1)\ldots(X-(r-1))]$ is the rth factorial moment.

Lemma 2. *Let $Y_1 = \sum_{i=1}^{m_1} I_i$ and $Y_2 = \sum_{j=1}^{m_2} I_j$, where (I_i) and (I_j) are indicator random variables. Then, for $r_1, r_2 \geq 1$,*

$$\mathbb{E}\left((Y_1)_{r_1}(Y_2)_{r_2}\right) = r_1! r_2! \sum_{\substack{1 \leq i_1 < \cdots < i_{r_1} \leq m_1 \\ 1 \leq j_1 < \cdots < j_{r_1} \leq m_2}} \mathbb{P}(I_{i_1} \cap \ldots \cap I_{i_{r_1}} \cap I_{j_1} \cap \ldots \cap I_{j_{r_2}}).$$

Proof. The proof of this statement was left as an exercise in [19]. Therefore, for sake of completeness, we include the proof. Notice that

$$\begin{aligned}
z_1^{Y_1} z_2^{Y_2} &= z_1^{\sum_{i=1}^{m_1} I_i} z_2^{\sum_{j=1}^{m_2} I_j} = \prod_{i=1}^{m_1} z_1^{I_i} \prod_{j=1}^{m_2} z_2^{I_j} \\
&= \prod_{i=1}^{m_1} (1 + (z_1 - 1))^{I_i} \prod_{j=1}^{m_2} (1 + (z_2 - 1))^{I_j} \\
&= \prod_{i=1}^{m_1} (1 + I_i(z_1 - 1)) \prod_{j=1}^{m_2} (1 + I_j(z_2 - 1)) \\
&= 1 + \sum_{r_1=1}^{m_1} \sum_{r_2=1}^{m_2} \left(\sum_{\substack{1 \leq i_1 < \cdots < i_{r_1} \leq m_1 \\ 1 \leq j_1 < \cdots < j_{r_1} \leq m_2}} \left(\prod_{i=1}^{r_1} I_i \prod_{j=1}^{r_2} I_j \right) \right) (z_1 - 1)^{r_1} (z_2 - 1)^{r_2} \\
&= 1 + \sum_{r_1=1}^{m_1} \sum_{r_2=1}^{m_2} (z_1 - 1)^{r_1} (z_2 - 1)^{r_2} \left(\sum_{\substack{1 \leq i_1 < \cdots < i_{r_1} \leq m_1 \\ 1 \leq j_1 < \cdots < j_{r_1} \leq m_2}} \left(\prod_{i=1}^{r_1} I_i \prod_{j=1}^{r_2} I_j \right) \right).
\end{aligned}$$

Thus,

$$\mathbb{E}(z_1^{Y_1} z_2^{Y_2}) = 1 + \sum_{r_1=1}^{m_1} \sum_{r_2=1}^{m_2} (z_1 - 1)^{r_2} (z_1 - 1)^{r_2} \left(\sum_{\substack{1 \leq i_1 < \cdots < i_{r_1} \leq m_1 \\ 1 \leq j_1 < \cdots < j_{r_1} \leq m_2}} \mathbb{P}(I_{i_1} \cap \ldots \cap I_{i_{r_1}} \cap I_{j_1} \cap \ldots \cap I_{j_{r_2}}) \right),$$

and,

$$\begin{aligned}\mathbb{E}[(Y_1)_{r_1}(Y_2)_{r_2}] &= \frac{d^{r_1}}{dz_1^{r_1}}\frac{d^{r_2}}{dz_2^{r_2}}\left(\mathbb{E}\left(z^{Y_1Y_2}\right)\right)\Big|_{z_1=z_2=1} \\ &= r_1!r_2!\left(\sum_{\substack{1\le i_1<\cdots<i_{r_1}\le m_1\\ 1\le j_1<\cdots<j_{r_1}\le m_2}}\mathbb{P}(I_{i_1}\cap\ldots\cap I_{i_{r_1}}\cap I_{j_1}\cap\ldots\cap I_{j_{r_2}})\right).\end{aligned}$$

□

In addition to these two lemmas, we will also need the following which was proven in [17].

Lemma 3. *Let*

$$J_{r,m} := \{1 \le j_1 < ... < j_r \le m : \ j_l \le j_{l+1} - k, \ \ \forall l = 1, 2, ..., r-1\}.$$

Then

$$\sum_{J_{r,m}}\left(\prod_{l=1}^{r} j_{r-l+1}\right) = \frac{(m+1)_{2r}}{2^r r!}.$$

3. Preliminaries

We are interested in symbols along the kth diagonal of staircase tableaux and in order to use Lemma 1, we will need to compute asymptotically

$$\mathbb{P}(\alpha^k_{j_1}, \alpha^k_{j_2}, \ldots, \alpha^k_{j_r}), \ \ 1 \le j_1 < \cdots < j_r \le n-k+1, \text{ and}$$
$$\mathbb{P}(x^k_{j_1}, x^k_{j_2}, \ldots, x^k_{j_r}), \ \ 1 \le j_1 < \cdots < j_r \le n-k+1$$

where the remaining $n-r-k+1$ boxes on the kth diagonal are not specified (i.e. may contain an α, β, or be empty). In this section, we will focus on the latter probability as this case will easily reduce to the α case.

To compute $\mathbb{P}(x^k_{j_1}, x^k_{j_2}, \ldots, x^k_{j_r})$, we will consider two cases: $m = 1$ and $m > 1$ where

$$m := \min\{l \ge 1 : \ j_l \le j_{l+1} - k\},$$

i.e. all symbols on the kth diagonal before the mth are within k columns of the subsequent symbol, but the mth is not. Notice that $j_2 - j_1 \le (k-2), \ldots, j_m - j_{m-1} \le (k-2)$ (since $j_l = j_{l+1} - k - 1$ is impossible by the rules of staircase tableaux) and thus,

$$j_m \le (m-1)(k-2) + j_1. \tag{3}$$

Now consider arbitrary $S \in \mathcal{S}_n$ such that $x^k_{j_1}, x^k_{j_2}, \ldots, x^k_{j_r}$. By removing the first $j_1 - 1$ columns (see Equation (2)) we may and do assume that $j_1 = 1$ but we must keep in mind that b now depends on n. We are interested in keeping track of certain symbols in the first $\widehat{k} := k + j_m - 1$ columns of S. It is important to note that since $j_m \le (m-1)(k-2)+1$ (by (3) and the assumption that $j_1 = 1$), $\widehat{k}$ does not depend on n, but rather on k and m which are both fixed. In addition, note that when $m = 1$, $\widehat{k} = k$.

Define a *restricted* symbol to be any β below a symbol (which by the rules of staircase tableaux cannot be an α) or any α to the right of a symbol (which cannot be a β). The symbols we are interested in are those which are restricted by $x^k_1, \ldots, x^k_{j_m}$. First of all, notice that there are m α's and m β's on the first diagonal which fit this description. In addition, there are many others which are not on the first diagonal.

In order to classify these symbols, let D be the broken line going vertically from β^1_1 to x^k_1 then horizontally from x^k_1 to $0^{k+1-j_2}_{j_2}$ then vertically from $0^{k+1-j_2}_{j_2}$ to $x^k_{j_2}$ etc., until the vertical segment reaches $x^k_{j_m}$ and then the last horizontal segment goes from $x^k_{j_m}$ to $\alpha^1_{k+j_m-1}$ (see Figure 2 for an example). Then the following definition and lemma allow us to keep track of the remaining restricted symbols.

Definition 3. *Define a symbol to be D-connected if it is not on the first diagonal and one of the following two conditions hold:*

1. *The symbol lies on D but is not on the kth diagonal.*
2. *There exists a symbol above or to the left that is D-connected or lies on D.*

Lemma 4. *Properties of D-connected symbols:*

1. *Any symbol in the same column as a D-connected α or the same row as a D-connected β is also D-connected (provided it is not on the first diagonal).*

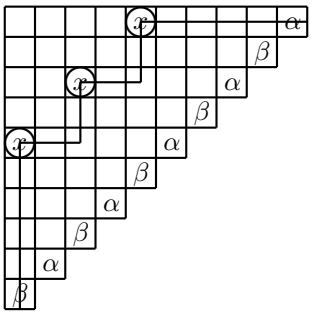

FIGURE 2. A staircase tableau with $j_1 = 1$, $j_2 = 3$, $j_3 = 5$, $m = 3$, $k = 6$, and $\widehat{k} = 10$. D is denoted by a black line and specified symbols on the kth diagonal are circled.

(2) *There are at most* $\widehat{k} - m - 1$ *D-connected symbols.*
(3) *Each D-connected symbol can be paired uniquely with an opposite symbol on the first diagonal.*

Proof. Condition (1) follows directly from the definition of D-connected symbols and the rules of staircase tableaux.

Condition (2) follows from the fact that D-connected symbols, by definition, only lie in the region bounded by D and the first diagonal. This region is contained within a subtableau of size $\widehat{k}$ and therefore, D contains at most $2\widehat{k}-1$ symbols. Since there are already m symbols on the kth diagonal and $\widehat{k}$ symbols on the first diagonal, there are at most $\widehat{k} - m - 1$ D-connected symbols.

The pairing from Condition (3) can be defined as follows: pair each D-connected β with the first diagonal α it restricts in the same row. Pair each D-connected α with the first diagonal β it restricts in the same column. This pairing is unique since there is only one β in any given row and only one α in any given column. □

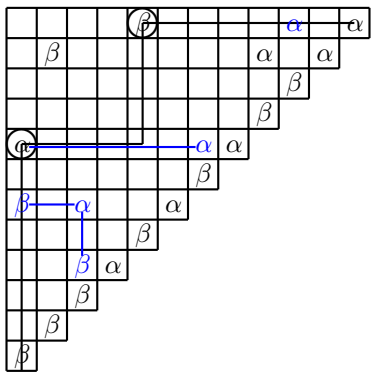

FIGURE 3. A staircase tableau of size 12 with $j_1 = 1$, $j_2 = 5$, $m = 2$, $k = 8$, and $\widehat{k} = 12$. D is denoted by a black line. There are five D-connected symbols (blue).

In subsequent sections, we will calculate $\mathbb{P}_{n,\alpha,\beta}(x_1^k, x_{j_2}^k, \ldots, x_{j_r}^k)$ by induction. To apply the inductive hypothesis, we will reduce an arbitrary tableau with r specified symbols on the kth diagonal to obtain a new, smaller tableau with $r - m$ specified symbols on the kth diagonal. This can be done by removing some (but not all) of the first $\widehat{k}$ columns and some of the last $\widehat{k}$ rows. In particular, we will remove the rows and columns which satisfy the following definition,

Definition 4. *A column (row) is completely restricted if any of the following conditions hold:*

(1) *The column (row) contains an α (β) on the kth diagonal and jth row for $j \in \{j_1, j_2, \ldots, j_r\}$.*
(2) *The column (row) contains a first diagonal α (β) which shares a row (column) with an β (α) on the kth diagonal and jth row for $j \in \{j_1, j_2, \ldots, j_r\}$.*
(3) *There exists a D-connected α (β) in that column (row).*
(4) *There exists a first diagonal α (β) in that column (row) which is paired with a D-connected symbol.*

In addition, if j_1 is the first column and contains a kth diagonal α, then the $(n-k)$th row is also completely restricted.

This somewhat lengthy definition was designed to encompass columns and rows in which every symbol in that column or row is restricted by x_1^k, $x_{j_2}^k, \ldots,$ or $x_{j_r}^k$. An additional property is that removing all completely restricted columns and rows preserves the staircase tableau shape (this can be deduced by definition and Lemma 4(3)). Figure 4 illustrates these properties.

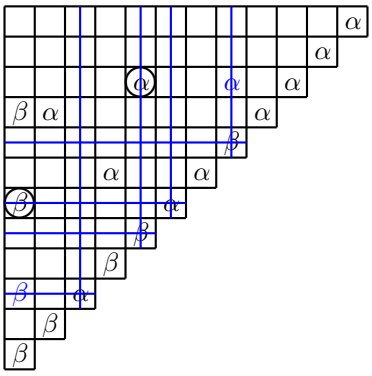

FIGURE 4. A staircase tableau of size 12 with $j_1 = 1$, $j_2 = 5$, $m = 2$, $k = 6$, $\widehat{k} = 10$ and 2 D-connected symbols (blue). Completely restricted columns and rows are indicated with a blue line. The different ways columns and rows can be completely restricted is illustrated (except the special case when there is an α_1^k).

The information that gets lost when removing all the completely restricted columns is the number of D-connected symbols and their particular arrangement within D. In order to keep track of this information, let $\mathcal{A}_{k,h}$ be the set of all possible arrangements of h D-connected symbols in a tableau of size k (note that D will be clear from the context). For example, $\mathcal{A}_{4,2} = \{\beta_1^2 \cap \beta_1^3, \beta_1^2 \cap \alpha_3^3, \beta_1^3 \cap \alpha_2^3, \beta_2^2 \cap \alpha_3^3, \beta_1^3 \cap \alpha_2^2, \alpha_2^3 \cap \alpha_3^2\}$. Note that $\mathcal{A}_{k,0} = \{\emptyset\}$ for all $k \geq 1$ as there are no arrangements of 0 symbols.

4. The asymptotic distribution of alphas on the kth diagonal

In this section, our goal is to compute the asymptotic distribution of α's along the kth diagonal. To do so, we will first compute $\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}^k, \ldots, \alpha_{j_r}^k)$ which requires a discussion on how to obtain a new, smaller tableau from an arbitrary tableau $S \in \mathcal{T}_{n,\alpha}$ where $\mathcal{T}_{n,\alpha} := \{S \in \mathcal{S}_{n,\alpha,\beta} : \ \alpha_1^k, \alpha_{j_2}^k, \ldots, \alpha_{j_r}^k\}$. We will do so in two cases.

4.1. **The case m=1.** Consider an arbitrary $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols. If $\mathcal{T}_{n,\alpha,h} := \{S \in \mathcal{S}_{n,\alpha,\beta} : \alpha_{j_2-h-2}^k, \ldots, \alpha_{j_r-h-2}^k\}$, define $T_S \in \mathcal{T}_{n-h-2,\alpha,h}$ to be the subtableau obtained from S by removing all completely restricted columns and rows. An illustration of this process is given in Figure 5.

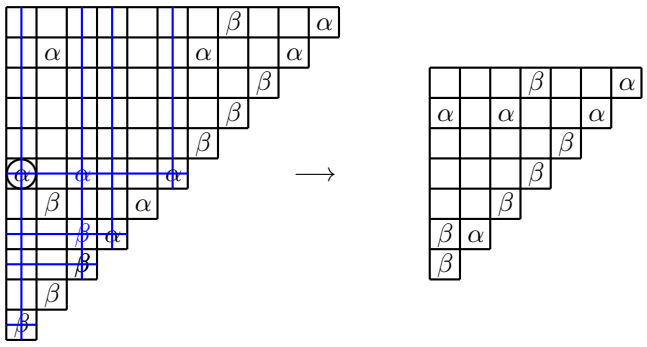

FIGURE 5. An example of the construction of $T_S \in \mathcal{T}_{7,\alpha,2}$ given a tableau $S \in T_{11,\alpha}$ in the case $j_1 = 1$, $m = 1$, $k = 6$.

This construction implies for any $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols

$$wt(S) = \alpha^{h+2}\beta^{h+1}wt(T_S) \tag{4}$$

since the weight of the deleted columns and rows is $\alpha^{h+2}\beta^{h+1}$ by Definition 4 and Lemma 4.

Now consider the following lemma,

Lemma 5. *When $m = 1$, there exists a bijection between $T_{n,\alpha}$ and triples (h, a, T) where $0 \le h \le k-2$, $a \in \mathcal{A}_{k,h}$ and $T \in \mathcal{T}_{n-h-2,\alpha,h}$.*

Proof. For arbitrary $S \in T_{n,\alpha}$, if h_S is the number of D-connected symbols of S, $a_S \in \mathcal{A}_{k,h}$ is the particular arrangement of those h_S D-connected symbols, and $T_S \in \mathcal{T}_{n-h_S-2,\alpha,h_S}$ is the subtableau of S obtained by removing all completely restricted rows and columns, then

$$F:\ S \mapsto (h_S, a_S, T_S)$$

explicitly defines such a bijection (see Figure 6 for an example).

To prove F is injective, consider $S_1, S_2 \in T_{n,\alpha}$ such that $S_1 \neq S_2$. This implies that there exists at least one box in S_1 which is different from S_2, say (i, j) (notice that this box cannot be $(n-k+1, 1)$ as both S_1 and S_2 have an α there). Without loss of generality, assume that box (i, j) in S_1 contains a α and box (i, j) in S_2 is either empty or contains a β. In addition, we can assume that $h_{S_1} = h_{S_2}$ and $a_{S_1} = a_{S_2}$ since otherwise it is clear that $F(S_1) \neq F(S_2)$.

With these assumptions, the α in S_1 is not D-connected. Therefore by Definition 3, it is not in row $n-k$ and there is no D-connected β in the same row. This is enough (in this context) to conclude that it is not in a completely restricted column or row (see Definition 4). Therefore, this α will appear in the tableau T_{S_1}. Since $a_{S_1} = a_{S_2}$, all the same rows and columns will be deleted in S_2 as in S_1 when obtaining the subtableaux T_{S_1} and T_{S_2}. Thus, the (i, j)th box in S_2 (which is empty or contains a β) will appear in T_{S_2} and in the same position as the α in T_{S_1}. Therefore, $T_{S_1} \neq T_{S_2}$.

Now to prove F is surjective, consider arbitrary triple (h, a, T) where $0 \le h \le k-2$, $a \in \mathcal{A}_{k,h}$, and $T \in \mathcal{T}_{n-h-2,\alpha,h}$. The following is a description of how to construct an $S \in \mathcal{T}_{n,\alpha}$ such that $F(S) = (h, a, T)$.

First, construct an (incomplete) staircase tableau of size n such α_1^k, α_k^1, and β_n^1. In addition, fill in the particular arrangement of D-connected symbols as described by a. Then insert T into boxes which are not on completely restricted columns or rows (T is inserted in order by columns, see Figure 7).

The number of completely restricted columns and rows are both $h+2$ in this context. Therefore, the number of boxes which are not on completely restricted columns or rows form the shape of a staircase tableaux of size $n-h-2$, which is exactly the size of T. In addition, inserting T will not violate the rules of staircase tableaux since α_1^k and D-connected symbols only restrict D-connected symbols (see Lemma 4(1)) and all of the columns and rows containing D-connected symbols are completely restricted (this can be deduced from Definition 4). Therefore, this construction formulates a staircase tableau, $S \in T_{n,\alpha}$.

□

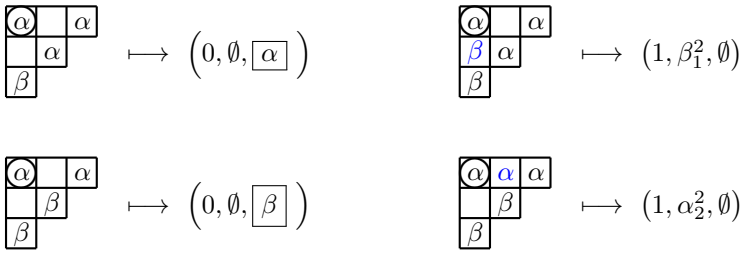

FIGURE 6. The map F from Lemma 5 when $n = 3$ and $k = 3$.

Now let

$$C_{k,h} := |\mathcal{A}_{k,h}|,$$

and note that $C_{k,h}$ depends on k and h but not on n (and $C_{k,0} = 1$). In addition, note that (4) holds for all S with h D-connected symbols regardless of the particular arrangement of those h D-connected

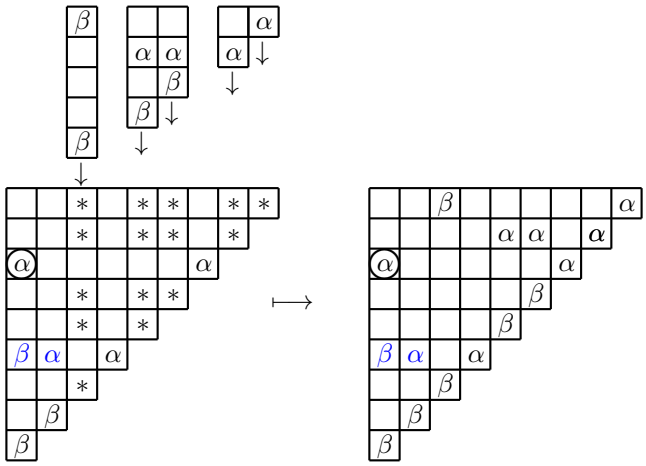

FIGURE 7. An example of the construction of $S \in \mathcal{T}_{9,\alpha}$ given the triple $(2, \beta_1^4 \cap \alpha_2^3, T)$ where $T \in \mathcal{T}_{5,\alpha,2}$ is the tableau being inserted (upper left). In this case, $j_1 = 1$ and $k = 7$. Boxes which are not on completely restricted columns or rows are denoted with an "$*$".

symbols. Therefore by Lemma 5 and Equation (4),

$$\sum_{S \in \mathcal{T}_{n,\alpha}} wt(S) = \sum_{h=0}^{k-2} C_{k,h} \alpha^{h+2} \beta^{h+1} \sum_{T \in \mathcal{T}_{n-h-2,\alpha,h}} wt(T).$$

By Definition 2, dividing the total number of weights by $Z_n(\alpha,\beta) = (\alpha\beta)^{h+2}(a+b+n-1)_{h+2} Z_{n-h-2}(\alpha,\beta)$ gives the following (in the case $m = 1$),

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1^k, \alpha_{j_2}^k, \ldots, \alpha_{j_r}^k) \quad = \quad \sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}} \mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-h-2}^k, \ldots, \alpha_{j_r-h-2}^k). \tag{5}$$

4.2. **The case m>1.** In this case, we only need an upper bound. Consider arbitrary $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols. For this section, let $\mathcal{T}_{n,\alpha,h} := \{S \in \mathcal{S}_{n,\alpha,\beta} : \alpha_{j_{m+1}-h-m}^k, \ldots, \alpha_{j_r-h-m}\}$ and then define $T_S \in \mathcal{T}_{n-h-m,\alpha,h}$ to be the subtableau obtained from S by removing all completely restricted columns and rows. An illustration of this process in the case $m > 1$ is given in Figure 8. This construction

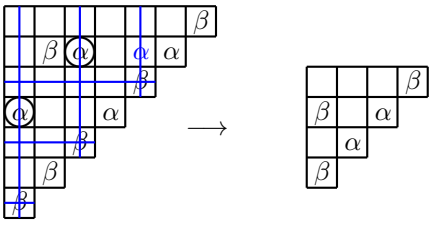

FIGURE 8. An example of the construction of $T_S \in \mathcal{T}_{4,\alpha,1}$ given a tableau $S \in T_{7,\alpha}$ in the case $j_1 = 1$, $j_2 = 3$, $m = 2$ and $k = 4$.

implies for any $S \in \mathcal{T}_{n,\alpha}$ with h D-connected symbols

$$wt(S) = \alpha^{h+m} \beta^{h+m} wt(T_S) \tag{6}$$

since the weight of the deleted columns and rows is $\alpha^{h+m}\beta^{h+m}$ by Definition 4 and Lemma 4. From this discussion, we have the following lemma whose proof is analogous to the first part of the proof of Lemma 5.

Lemma 6. *When $m > 1$, there exists a injective map between $\mathcal{T}_{n,\alpha}$ and triples (h, a, T) where $0 \le h \le \widehat{k} - m - 1$, $a \in \mathcal{A}_{\widehat{k},h}$ and $T \in \mathcal{T}_{n-h-m,\alpha,h}$.*

It is perhaps worthwhile to note that the map from Lemma 6 would be a bijection if the set of tableaux $\mathcal{T}_{n-h-m,\alpha,h}$ was further restricted having some α's on the first diagonal. However, a bijection is not necessary in this case and we prefer to work with the simpler set $\mathcal{T}_{n-h-m,\alpha,h}$.

Therefore, by Lemma 6 and Equation (6)

$$\sum_{S\in\mathcal{T}_{n,\alpha}} wt(S) \leq \sum_{h=0}^{\widehat{k}-m-1} C_{\widehat{k},h}\alpha^{h+m}\beta^{h+m} \sum_{T\in\mathcal{T}_{n-h-m,\alpha,h}} wt(T).$$

Notice that $C_{\widehat{k},h}$ does not depend on n since $\widehat{k}$ only depends on k and m per the discussion in Section 3. Dividing by $Z_n(\alpha,\beta) = (\alpha\beta)^{h+m}(a+b+n-1)_{h+m}Z_{n-h-m}(\alpha,\beta)$, we obtain (for the case $m>1$),

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1^k,\alpha_{j_2}^k,\ldots,\alpha_{j_r}^k) &\leq \sum_{h=0}^{\widehat{k}-m-1} \frac{C_{\widehat{k},h}}{(n+a+b-1)_{h+m}}\mathbb{P}_{n-h-m,\alpha,\beta}(\alpha^k_{j_{m+1}-h-m},\ldots,\alpha^k_{j_r-h-m}) \\ &= O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m-1}\mathbb{P}_{n-h-m,\alpha,\beta}(\alpha^k_{j_{m+1}-h-m},\ldots,\alpha^k_{j_r-h-m}). \qquad (7)\end{aligned}$$

4.3. **The distribution of alphas.** We can now give the asymptotic joint distribution of the α's on the kth diagonal. Note that since we will be dealing exclusively with the kth diagonal through the remainder of this section we will drop the superscript k.

Theorem 7. *Let* $1\leq j_1<\ldots<j_r\leq n-k+1$. *If*

$$j_l \leq j_{l+1}-k,\ \ \forall l=1,2,\ldots,r-1$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1},\ldots,\alpha_{j_r}) = \prod_{l=1}^{r}\frac{j_{r-l+1}}{n^2} + O\left(\frac{1}{n^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1},\ldots,\alpha_{j_r}) = O\left(\frac{1}{n^r}\right).$$

Proof. In this proof, b will sometimes depend on n, so we will first prove that if $j_l\leq j_{l+1}-k,\ \forall\ l=1,2,\ldots,r-1$, then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1},\ldots,\alpha_{j_r}) = \prod_{l=1}^{r}\frac{b+j_{r-l+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1},\ldots,\alpha_{j_r}) = O\left(\frac{1}{(n+b)^r}\right).$$

This statement reduces to the desired claim when b is fixed.

In fact, from this statement, it suffices to prove the theorem for $j_1=1$ and $b=O(n)$. For if $j_1>1$, by Equation (2), we can consider $\mathbb{P}_{n-j_1+1,\alpha,\widehat{\beta}}(\alpha_1,\alpha_{j_2-j_1+1}\ldots,\alpha_{j_r-j_1+1})$ where $\widehat{b}=b+j_1-1$ and the conclusion will follow from the $j_1=1$ case.

Therefore, without loss of generality, assume $j_1=1$ and $b=O(n)$. Then the proof follows by induction on r. When $r=1$, apply Equation (5)

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1) = \sum_{h=0}^{k-2} C_{k,h}\frac{b}{(n+a+b-1)_{h+2}}.$$

Since $C_{k,0}=1$ and $C_{k,h}$ does not depend on n for any h and k,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1) &= \frac{b}{(n+a+b-1)_2} + O\left(\frac{1}{(n+b)^2}\right)\sum_{h=1}^{k-2}C_{k,h} \\ &= \frac{b}{(n+b)^2} + O\left(\frac{1}{(n+b)^2}\right).\end{aligned}$$

Assume the statement holds for integers up to $r-1$ and then consider the following two cases.

Case 1. $m=1$.

Applying Equation (5),

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) &= \sum_{h=0}^{k-2} C_{k,h}\frac{b}{(n+a+b-1)_{h+2}}\mathbb{P}_{n-2,\alpha,\beta}(\alpha_{j_2-h-2},\ldots,\alpha_{j_r-h-2}) \\ &= \frac{b}{(n+a+b-1)_2}\mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-2},\ldots,\alpha_{j_r-2}) + \\ &\quad +O\left(\frac{1}{(n+b)^2}\right)\sum_{h=1}^{k-2}\mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-h-2},\ldots,\alpha_{j_r-h-2}).\end{aligned}$$

By the induction hypothesis, if $j_l \leq j_{l+1}-k$ for all $l=2,\ldots,r-1$,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) &= \frac{b}{(n+a+b-1)_2}\left(\prod_{l=1}^{r-1}\frac{b+j_{r-l+1}}{(n+b)^2}+O\left(\frac{1}{(n+b)^r}\right)\right) \\ &\quad +O\left(\frac{1}{(n+b)^2}\right)O\left(\frac{1}{(n+b)^{r-1}}\right) \\ &= \prod_{l=1}^{r}\frac{b+j_{r-l+1}}{(n+b)^2}+O\left(\frac{1}{(n+b)^{r+1}}\right).\end{aligned}$$

On the other hand, if for some $2\leq l\leq r-1$, $j_l > j_{l+1}-k$ then by the induction hypothesis again,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) &= \frac{b}{(n+b)^2}\left(O\left(\frac{1}{(n+b)^{r-1}}\right)\right)+O\left(\frac{1}{(n+b)^2}\right)O\left(\frac{1}{(n+b)^{r-1}}\right) \\ &= O\left(\frac{1}{(n+b)^r}\right).\end{aligned}$$

Case 2. $m>1$.

Notice that the cases $j_l=k$, $l\leq m$ are impossible by the rules of staircase tableaux and thus have probability zero.

For all other cases, by Equation (7)

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) = O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m-1}\mathbb{P}_{n-h-m,\alpha,\beta}(\alpha_{j_{m+1}-h-m},\ldots,\alpha_{j_r-h-m}).$$

By the induction hypothesis,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1,...,\alpha_{j_r}) &= O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m-1}O\left(\frac{1}{(n+b)^{r-m}}\right) \\ &= O\left(\frac{1}{(n+b)^r}\right).\end{aligned}$$

Combining Cases 1 and 2 gives the result. □

We can now give the asymptotic distribution of the number of α's on the kth diagonal.

Theorem 8. *Let $A_n^{(k)}$ be the number of α's on the kth diagonal of a random weighted staircase tableau. Then, as $n\to\infty$,*

$$A_n^{(k)} \xrightarrow{d} Pois\left(\frac{1}{2}\right).$$

Proof. By [2, Theorem 22, Chapter 1] and Lemma 1, it suffices to show that for all r,

$$\sum_{1\leq j_1<\ldots<j_r\leq n-k+1}\mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r})\to\frac{1}{2^r r!}$$

as $n\to\infty$. Write

$$\sum_{1\leq j_1<\ldots<j_r\leq n-k+1}\mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r}) = \sum_{J}\mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r})+\sum_{J^c}\mathbb{P}(\alpha_{j_1},\ldots,\alpha_{j_r}),$$

where

$$J := \{1 \le j_1 < \cdots < j_r \le n-k+1 : \ \forall\, l = 1, \dots, r-1;\ j_{l+1} - j_l \ge k\}$$

and

$$J^c := \{1 \le j_1 < \cdots < j_r \le n-k+1 : \ \exists\, l = 1, \dots, r-1;\ j_{l+1} - j_l < k\}.$$

In addition, recall that

$$J_{r,m} := \{1 \le j_1 < ... < j_r \le m : \ j_l \le j_{l+1} - k, \ \ \forall l = 1, 2, ..., r-1\}.$$

Then by Theorem 7,

$$\sum_J \mathbb{P}(\alpha_{j_1}, \dots, \alpha_{j_r}) = \sum_J \left(\prod_{l=1}^{r} \frac{j_{r-l+1}}{n^2} + O\left(\frac{1}{n^{r+1}}\right)\right)$$
$$= \sum_J \prod_{l=1}^{r} \frac{j_l}{n^2} + \binom{n-k+1}{r} \cdot O\left(\frac{1}{n^{r+1}}\right) = \sum_{J_{r,n-k+1}} \prod_{l=1}^{r} \frac{j_l}{n^2} - \sum_{J_{r,n-k+1} \setminus J} \prod_{l=1}^{r} \frac{j_l}{n^2} + O\left(\frac{1}{n}\right).$$

If $(j_1, \dots, j_r) \in J_{r,n-k+1} \setminus J$ then there exists an l such that $j_{l+1} - j_l < k$ and thus this set has $O\left(\binom{n-k+1}{r-1}\right)$ elements. By this fact and Lemma 3,

$$\sum_J \mathbb{P}(\alpha_{j_1}, \dots, \alpha_{j_r}) = \frac{1}{2^r r!} + O\left(\binom{n-k+1}{r-1} \cdot \frac{n^r}{n^{2r}}\right) + O\left(\frac{1}{n}\right) = \frac{1}{2^r r!} + O\left(\frac{1}{n}\right).$$

Finally, by Theorem 7,

$$\sum_{J^c} \mathbb{P}(\alpha_{j_1}, \dots, \alpha_{j_r}) = O\left(\binom{n-k+1}{r-1} \frac{1}{n^r}\right) = O\left(\frac{1}{n}\right).$$

Combining these expressions and taking the limit as $n \to \infty$ completes the proof. □

Corollary 9. *Let $B_n^{(k)}$ to be the number of β's on the kth diagonal of a random weighted staircase tableau. Then, as $n \to \infty$,*

$$B_n^{(k)} \xrightarrow{d} Pois\left(\frac{1}{2}\right).$$

Proof. This follows from the involution discussed in Section 2. □

5. The asymptotic distribution of symbols on the kth diagonal

In this section, our goal is to compute the asymptotic distribution of the total number of symbols along the kth diagonal. It turns out that α and β are asymptotically independent. As in Section 4, we will first compute $\mathbb{P}_{n,\alpha,\beta}(x^k_{j_1}, \dots, x^k_{j_r})$ which requires a discussion on how to obtain a new, smaller tableau from an arbitrary tableau $S \in \mathcal{T}_{n,x}$ where $\mathcal{T}_{n,x} := \{S \in \mathcal{S}_n : x^k_1, x^k_{j_2}, \dots, x^k_{j_r}\}$. We will do so in two cases.

5.1. **The case m=1.** If the first symbol is an α, then the probability follows easily from the discussion in Section 4 and we obtain the following equation analogous to (5) for the case $m = 1$,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha^k_1, x^k_{j_2}, \dots, x^k_{j_r}) = \sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}} \mathbb{P}_{n-h-2}(x^k_{j_2-h-2}, \dots, x^k_{j_r-h-2}). \tag{8}$$

When the first symbol is a β, consider arbitrary $S \in \mathcal{T}_{n,x}$ with h D-connected symbols. Notice that there are $k-h-2$ empty boxes in the first column between β^1_1 and β^k_1 which are not on a completely restricted row. These empty boxes will remain after removing all completely restricted columns and rows. Therefore, let $\beta^i_j(0_l)$ denote the event that there is a β in column j and diagonal i and there are l empty boxes directly above β in the same column and define $\mathcal{T}_{n,x,h,0} := \{T \in \mathcal{S}_n : \beta^1_1(0_{k-h-2}), x^k_{j_2-h-1}, \dots, x^k_{j_r-h-1}\}$. From this discussion, the subtableau, say T_S, obtained from S by removing all completely restricted columns and rows is a tableau from the set $\mathcal{T}_{n-h-1,x,h,0}$. An illustration of this process in the case $m = 1$ is given in Figure 9.

This construction implies that for all $S \in \mathcal{T}_{n,x}$ with h D-connected symbols

$$wt(S) = \alpha^{h+1}\beta^{h+1} wt(T_S) \tag{9}$$

since the weight of the deleted columns and rows is $\alpha^{h+1}\beta^{h+1}$ by Definition 4 and Lemma 4.

Now consider the following lemma,

Lemma 10. *When $m = 1$, there exists a bijection between $\mathcal{T}_{n,x}$ and triples (h, a, T) where $0 \le h \le k-2$, $a \in \mathcal{A}_{k,h}$ and $T \in \mathcal{T}_{n-h-1,x,h,0}$.*

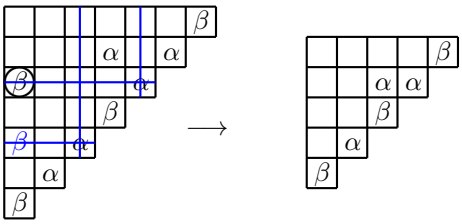

FIGURE 9. An example of the construction of $T_S \in \mathcal{T}_{5,x,1,0}$ given a tableau $S \in \mathcal{T}_{7,x}$ in the case $j_1 = 1$, $m = 1$, and $k = 5$.

Proof. The proof is analogous to the proof of Lemma 5 although the construction of an $S \in \mathcal{T}_{n,x}$ given a triple (h, a, T) should be slightly modified. In particular, construct a (incomplete) staircase tableau of size n such that β_1^k and α_k^1 and insert the particular arrangement of D-connected symbols as depicted by a (as before). Then insert T as before. Figure 10 illustrates this process. The proof that this constructs a staircase tableau $S \in \mathcal{T}_{n,x}$ follows similarly to the proof of Lemma 5. □

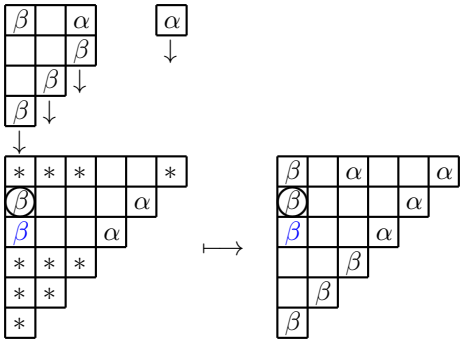

FIGURE 10. An example of the construction of $S \in \mathcal{T}_{6,x}$ given the triple $(1, \beta_1^4, T)$ where $T \in \mathcal{T}_{4,x,1}$ is the tableau being inserted (upper left). In this case, $j_1 = 1$ and $k = 5$. Boxes which are not on completely restricted columns and rows are denoted with an "$*$".

Furthermore, notice that a tableau with $\beta_1^1(0_{k-h-2})$ can be mapped to a tableau with β_{k-h-1}^1 in such a way which preserves the weight. The following lemma will make this clear.

Lemma 11. *There exists a weight preserving bijection between* $\mathcal{T}_{n,x,h,0}$ *and* $\mathcal{T}_{n,x,h} := \{T \in \mathcal{S}_n : \beta_{k-h-1}^1, x_{j_2-h-1}^k, \ldots, x_{j_r-h-1}^k\}$.

Proof. For arbitrary tableau $T \in \mathcal{T}_{n,x,h,0}$, consider the subtableau $T[n-k+h+2, 1]$ whose first column contains a β on the first diagonal and all empty boxes above. Invert this first column into a row and place it on top of the remaining piece of the subtableau $T[n-k+h+2, 1]$. The first row will now have all empty boxes except a β on the first diagonal. Notice that the rules of staircase tableaux are still preserved and that this is still a tableaux of size $k-h-1$.

Now consider the remaining boxes in the first $k-h-1$ columns which are not in the subtableau $T[n-k+h+2, 1]$. The tableaux of size $k-h-1$ below this rectangle has been rearranged and consequently, the columns above must also be rearranged to preserve the rules of staircase tableaux. First notice that these remaining boxes constitute a rectangle of size $k-h-1 \times n-k+h+1$ and the only columns which have symbols are those which are in the same column as a first diagonal β, say β-columns. Therefore, move the first β-column above the first diagonal box which contains a β (if such a move is necessary), and continue in this manner until all of the β-columns are above β diagonal boxes and in the same order as the columns were originally. These moves prevent symbols from being above an α, and the symbols in the first $n-k+h+1$ rows remain in the same order so that there is no symbol to the left of a β. Therefore, this maps T to a tableau of the same size and the same weight with a β_{k-h-1}^1. See Figure 11 for an illustration of this map.

The inverse map can be easily be constructed by reversing these steps and thus, the composition of these two maps is the identity map. □

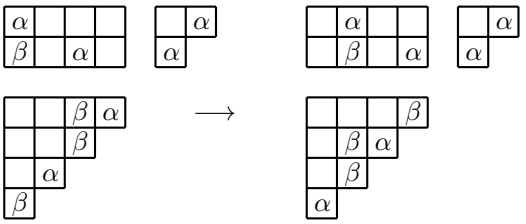

FIGURE 11. An illustration of the map described in Lemma 11 with $\beta_1^1(0_3)$.

Therefore, by Lemma 10, Lemma 11, and Equation (9)

$$\sum_{S\in\mathcal{T}_{n,x}} wt(S) = \sum_{h=0}^{k-2} C_{k,h}\alpha^{h+1}\beta^{h+1} \sum_{T\in\mathcal{T}_{n-h-1,x,h}} wt(T).$$

Now to simplify the set on the right, consider (for now) tableaux with a fixed number of D-connected symbols, say h, and a fixed arrangement of those symbols, say a. In this case, dividing the total number of weights, $\alpha^{h+1}\beta^{h+1}\sum_{T\in\mathcal{T}_{n-h-1,x,h}} wt(T)$, by $Z_n(\alpha,\beta) = (\alpha\beta)^{h+1}(a+b+n-1)_{h+1}Z_{n-h-1}(\alpha,\beta)$ gives the following,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\beta_1^k, x_{j_2}^k,\ldots,x_{j_r}^k,h,a) &= \frac{1}{(a+b+n-1)_{h+1}}\mathbb{P}_{n-h-1,\alpha,\beta}(\beta_{k-h-1}^1, x_{j_2-h-1}^k,\ldots,x_{j_r-h-1}^k)\\
&= \frac{1}{(a+b+n-1)_{h+1}}\mathbb{P}_{n-h-1,\alpha,\beta}(x_{j_2-h-1}^k,\ldots,x_{j_r-h-1}^k)\\
&\quad -\frac{1}{(a+b+n-1)_{h+1}}\mathbb{P}_{n-h-1,\alpha,\beta}(\alpha_{k-h-1}^1, x_{j_2-h-1}^k,\ldots,x_{j_r-h-1}^k).
\end{aligned} \tag{10}$$

Now, to compute the probability of the event $\{\alpha_{k-h-1}^1, x_{j_2-h-1}^k,\ldots,x_{j_r-h-1}^k\}$, delete the first $k-h-2$ columns using (2), defining $\widetilde{b} := b+k-h-2$, $\widetilde{n} := n-k+h+2$, and $\widetilde{j} := j-k+h+2$. Then,

$$\mathbb{P}_{n-h-1,\alpha,\beta}(\alpha_{k-h-1}^1, x_{j_2-h-1}^k,\ldots,x_{j_r-h-1}^k) = \mathbb{P}_{\widetilde{n}-h-1,\alpha,\widetilde{\beta}}(\alpha_1^1, x_{\widetilde{j}_2-h-1}^k,\ldots,x_{\widetilde{j}_r-h-1}^k).$$

Adding the weights of the tableaux such that $\{\alpha_1^1, x_{\widetilde{j}_2-h-1}^k,\ldots,x_{\widetilde{j}_r-h-1}^k\}$ and letting $\mathcal{T}_{n,\widetilde{x},h} := \{S\in\mathcal{S}_{n,\alpha,\widetilde{\beta}} : x_{\widetilde{j}_2-h-2}^k,\ldots,x_{\widetilde{j}_r-h-2}^k\}$, we obtain

$$\alpha\sum_{T\in\mathcal{T}_{\widetilde{n}-h-2,\widetilde{x}}} wt(T).$$

Dividing by $Z_{\widetilde{n}-h-1}(\alpha,\widetilde{\beta}) = (\alpha\widetilde{\beta})(a+\widetilde{b}+\widetilde{n}-h-3)Z_{\widetilde{n}-h-2}(\alpha,\widetilde{\beta})$,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\alpha_1^1, x_{\widetilde{j}_2-h-1}^k,\ldots,x_{\widetilde{j}_r-h-1}^k) &= \frac{\widetilde{b}}{(a+\widetilde{b}+\widetilde{n}-h-3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}(x_{\widetilde{j}_2-h-2}^k,\ldots,x_{\widetilde{j}_r-h-2}^k)\\
&= \frac{b+k-h-2}{(a+b+n-h-3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}(x_{\widetilde{j}_2-h-2}^k,\ldots,x_{\widetilde{j}_r-h-2}^k).
\end{aligned}$$

Therefore, returning to Equation (10),

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\beta_1^k, x_{\widetilde{j}_2-h-1}^k,\ldots,x_{\widetilde{j}_r-h-1}^k,h,a) &= \frac{1}{(a+b+n-1)_{h+1}}\mathbb{P}_{n-h-1}(x_{j_2-h-1}^k,\ldots,x_{j_r-h-1}^k)\\
&\quad -\frac{1}{(a+b+n-1)_{h+1}}\frac{b+k-h-2}{(a+b+n-h-3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}(x_{\widetilde{j}_2-h-2}^k,\ldots,x_{\widetilde{j}_r-h-2}^k).
\end{aligned}$$

Now if we consider tableaux with any number of D-connected symbols and any arrangement of those D-connected symbols, we obtain our desired calculation for the case $m=1$,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\beta_1^k, x_{j_2}^k,\ldots,x_{j_r}^k) = \sum_{h=0}^{k-2} C_{k,h}\Bigg(&\frac{1}{(a+b+n-1)_{h+1}}\mathbb{P}_{n-h-1}\left(x_{j_2-h-1}^k,\ldots,x_{j_r-h-1}^k\right)\\
&-\frac{b+k-h-2}{(a+b+n-1)_{h+1}(a+b+n-h-3)}\mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-h-2}^k,\ldots,x_{\widetilde{j}_r-h-2}^k\right)\Bigg).
\end{aligned} \tag{11}$$

5.2. **The case m>1.** In this case, we need only give an upper bound. Within $x_1^k, \ldots, x_{j_m}^k$, say there are l α's and $m-l$ β's for some l. In addition, for this section let $\mathcal{T}_{n,x,h} := \{S \in \mathcal{S}_n : x^k_{j_{m+1}-h-m}, \ldots, x^k_{j_r-h-m}\}$. For arbitrary $S \in \mathcal{T}_{n,x}$ with h D-connected symbols let $T_S \in \mathcal{T}_{n-h-m,x,h}$ to be the subtableau obtained from S by removing all completely restricted columns and rows. This construction implies for any $S \in \mathcal{T}_{n,x}$ with h D-connected symbols

$$wt(S) = \alpha^{h+m}\beta^{h+m} wt(T_S) \tag{12}$$

since the weight of the deleted columns and rows is $\alpha^{h+m}\beta^{h+m}$ by Definition 4 and Lemma 4. From this discussion, we have the following lemma whose proof is analogous to first part of the proof of Lemma 5.

Lemma 12. *When $m > 1$, there exists a injective map between $\mathcal{T}_{n,x}$ and triples (h, a, T) where $0 \le h \le \widehat{k} - m - 1$, $a \in \mathcal{A}_{\widehat{k},h}$ and $T \in \mathcal{T}_{n-h-m,x,h}$.*

Therefore, by Lemma 12

$$\sum_{S\in\mathcal{T}_{n,x}} wt(S) = \sum_{h=0}^{\widehat{k}-m-1} C_{\widehat{k},h}\alpha^{h+m}\beta^{h+m} \sum_{T\in\mathcal{T}_{n-h-m,x,h}} wt(T).$$

Dividing by $Z_n(\alpha,\beta) = (\alpha\beta)^{h+m}(a+b+n-1)_{h+m} Z_{n-h-m}(\alpha,\beta)$ and recalling that $C_{\widehat{k},h}$ does not depend on n, we obtain (for the case $m > 1$),

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(x_1^k, x_{j_2}^k, \ldots, x_{j_r}^k) &\le \sum_{h=0}^{\widehat{k}-m-1} \frac{C_{\widehat{k},h}}{(a+b+n-1)_{h+m}} \mathbb{P}_{n-h-m,\alpha,\beta}(x^k_{j_{m+1}-h-m}, \ldots, x^k_{j_r-h-m}) \\
&= O\left(\frac{1}{(n+b)^m}\right) \sum_{h=0}^{\widehat{k}-m-1} \mathbb{P}_{n-h-m,\alpha,\beta}(x^k_{j_{m+1}-h-m}, \ldots, x^k_{j_r-h-m}). \qquad (13)
\end{aligned}$$

5.3. **The joint distribution of symbols.** We can now give the asymptotic joint distribution of symbols on the kth diagonal. In fact, on the kth diagonal, symbols are asymptotically independent. To show this, we will first compute the probability of the event $\{x_{j_1}, \ldots, x_{j_{r_1+r_2}}\}$, where $0 \le r_1, r_2$ and $\{x_{j_1}, \ldots, x_{j_{r_1+r_2}}\}$ is a particular sequence of α's and β's. More specifically, we will say that there are r_1 α's and r_2 β's using the notation $\alpha_{j_1^a}, \ldots, \alpha_{j_{r_1}^a}$, $1 \le j_1^a < \cdots < j_{r_1}^a \le (n-k+1)$ and $\beta_{j_1^b}, \ldots, \beta_{j_{r_1}^b}$, $1 \le j_1^b < \cdots < j_{r_2}^b \le (n-k+1)$, for the ordering of α's and β's separately. As before, we will drop the superscript k.

Theorem 13. *For all $1 \le j_1 < \cdots < j_{r_1+r_2} \le (n-k+1)$. If,*

$$j_{l+1} - j_l \ge k, \forall l = 1, \ldots, r_1 + r_2 - 1$$

then

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_{r_1+r_2}}) &= \left(\prod_{l_1=1}^{r_1} \mathbb{P}_n\left(\alpha_{j^a_{r_1-l_1+1}}\right)\right)\left(\prod_{l_2=1}^{r_2} \mathbb{P}_n\left(\beta_{j^b_{r_2-l_2+1}}\right)\right) + O\left(\frac{1}{n^{r_1+r_2+1}}\right) \\
&= \left(\prod_{l_1=1}^{r_1} \frac{j^a_{r_1-l_1+1}}{n^2}\right)\left(\prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{n^2}\right) + O\left(\frac{1}{n^{r_1+r_2+1}}\right).
\end{aligned}$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(x_{j_1}, \ldots, x_{j_{r_1+r_2}}) = O\left(\frac{1}{n^{r_1+r_2}}\right).$$

Proof. As in Theorem 7, it suffices to prove the theorem for $j_1 = 1$ by allowing b to depend on n. Thus, we will first prove that if $j_{l+1} - j_l \ge k, \forall l = 1, \ldots, r_1 + r_2 - 1$, then

$$\mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) = \left(\prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1+1}}{(n+b)^2}\right)\left(\prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2})\right) + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(x_1, \ldots, x_{j_{r_1+r_2}}) = O\left(\frac{1}{(n+b)^{r_1+r_2}}\right).$$

This statement reduces to the desired claim when b is fixed.

The proof will be by induction on $r_1 + r_2$. If $r_1 + r_2 = 1$, then $j_1 = j_1^a$ or $j_1 = j_1^b$ and in either case the conclusion is trivial due to Equation (1).

Assume the statement holds for integers up to $r_1 + r_2 - 1$ and consider the following two cases:

Case 3. $m = 1$.

This case will be broken up into two possibilities:

Case 3.1. $j_1 = j_1^a$

Applying Equation (8) and recalling $b = O(n)$ and $C_{k,h}$ does not depend on n,

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}} \mathbb{P}_{n-h-2}(x_{j_2-h-2}, \ldots, x_{j_{r_1+r_2}-h-2}) \\ &= \frac{b}{(n+a+b-1)_2} \mathbb{P}_{n-2}(x_{j_2-2}, \ldots, x_{j_{r_1+r_2}-2}) \\ &\quad + O\left(\frac{1}{(n+b)^2}\right) \sum_{h=1}^{k-2} \mathbb{P}_{n-h-2}(x_{j_2-h-2}, \ldots, x_{j_{r_1+r_2}-h-2}).\end{aligned}$$

By the induction hypothesis, if $j_l \leq j_{l+1} - k$ for all $l = 1, \ldots, r_1 + r_2 - 1$ then this is

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \frac{b}{(n+a+b-1)_2} \left(\prod_{l_1=1}^{r_1-1} \frac{b + j^a_{r_1-l_1-1}}{(n+b)^2} \prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2}}\right)\right) \\ &\quad + O\left(\frac{1}{(n+b)^2}\right) O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right) \\ &= \prod_{l_1=1}^{r_1} \frac{b + j^a_{r_1-l_1-1}}{(n+b)^2} \prod_{l_2=1}^{r_2} \frac{n - j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right).\end{aligned}$$

On the other hand, if for some $2 \leq l \leq r-1, j_l > j_{l+1} - k$ then by the induction hypothesis again, this is

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\alpha_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \frac{b}{(n+a+b-1)_2} O\left(\frac{1}{(n+b)^{r_1+r_2}}\right) + O\left(\frac{1}{(n+b)^2}\right) O\left(\frac{1}{(n+b)^{r_1+r_2}}\right) \\ &= O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)\end{aligned}$$

Case 3.2. $j_1 = j_1^b$.

Applying Equation (11),

$$\begin{aligned}\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \sum_{h=0}^{k-2} C_{k,h} \Bigg(\frac{1}{(a+b+n-1)_{h+1}} \mathbb{P}_{n-h-1}\left(x_{j_2-h-1}, \ldots, x_{j_r-h-1}\right) \\ &\quad - \frac{b+k-h-2}{(a+b+n-1)_{h+1}(a+b+n-h-3)} \mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}\right) \Bigg) \\ &= \frac{1}{n+b} \mathbb{P}_{n-2}(x_{j_2-2}, \ldots, x_{j_r-2}) - \frac{b}{(n+b)^2} \mathbb{P}_{\widetilde{n}-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-2}, \ldots, x_{\widetilde{j}_r-2}\right) \\ &\quad + O\left(\frac{1}{(n+b)^2}\right) \sum_{h=0}^{k-2} C_{k,h} \mathbb{P}_{n-h-1}(x_{j_2-h-1}, \ldots, x_{j_r-h-1}) \\ &\quad + O\left(\frac{1}{(n+b)^2}\right) \sum_{h=0}^{k-2} C_{k,h} \mathbb{P}_{\widetilde{n}-h-2,\alpha,\widetilde{\beta}}\left(x_{\widetilde{j}_2-h-2}, \ldots, x_{\widetilde{j}_r-h-2}\right).\end{aligned}$$

By the induction hypothesis, if $j_l \leq j_{l+1} - k$ for all $l = 1, \ldots, r_1 + r_2 - 1$ then this is

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \frac{1}{n+b}\left(\prod_{l_1=1}^{r_1}\frac{b+j^a_{r_1-l_1+1}}{(n+b)^2}\prod_{l_2=1}^{r_2-1}\frac{n-j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)\right) \\
&\quad -\frac{b}{(n+b)^2}\left(\prod_{l_1=1}^{r_1}\frac{b+j^a_{r_1-l_1+1}}{(n+b)^2}\prod_{l_2=1}^{r_2-1}\frac{n-j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)\right) \\
&\quad +O\left(\frac{1}{(n+b)^2}\right)\left(O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right)\right) \\
&= \frac{n}{(n+b)^2}\left(\prod_{l_1=1}^{r_1}\frac{b+j^a_{r_1-l_1+1}}{(n+b)^2}\prod_{l_2=1}^{r_2-1}\frac{n-j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right)\right) \\
&\quad +O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right) \\
&= \prod_{l_1=1}^{r_1}\frac{b+j^a_{r_1-l_1+1}}{(n+b)^2}\prod_{l_2=1}^{r_2}\frac{n-j^b_{r_2-l_2+1}}{(n+b)^2} + O\left(\frac{1}{(n+b)^{r_1+r_2+1}}\right).
\end{aligned}$$

On the other hand, if for some $2 \leq l \leq r-1, j_l > j_{l+1} - k$ then by the induction hypothesis again,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(\beta_1, x_{j_2}, \ldots, x_{j_{r_1+r_2}}) &= \frac{1}{n+b}O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right) - \frac{b}{(n+b)^2}O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right) \\
&\quad +O\left(\frac{1}{(n+b)^2}\right)O\left(\frac{1}{(n+b)^{r_1+r_2-1}}\right) \\
&= \left(\frac{1}{(n+b)^{r_1+r_2}}\right).
\end{aligned}$$

Case 4. $m > 1$.

By Equation (13),

$$\mathbb{P}_{n,\alpha,\beta}(x_1, \ldots, x_{j_{r_1+r_2}}) = O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m}\mathbb{P}_{n-h-m-l}(x_{j_{m+1}-h-m}, \ldots, x_{j_{r_1+r_2}-h-m}).$$

By the induction hypothesis,

$$\begin{aligned}
\mathbb{P}_{n,\alpha,\beta}(x_1, \ldots, x_{j_{r_1+r_2}}) &= O\left(\frac{1}{(n+b)^m}\right)\sum_{h=0}^{\widehat{k}-m-1}O\left(\frac{1}{(n+b)^{r_1+r_2-m}}\right) \\
&= O\left(\frac{1}{(n+b)^{r_1+r_2}}\right).
\end{aligned}$$

□

We can now give the asymptotic distribution of the number of symbols on the kth diagonal. As before, define A_n and B_n to be the number of α's and β's respectively on the kth diagonal, i.e. $A_n := \sum_{j=1}^{n-k+2} I_{\alpha_j}$ and $B_n := \sum_{j=1}^{n-k+2} I_{\beta_j}$. Also, let

$$J = \{1 \leq j^a_1 < \cdots < j^a_{r_1} \leq (n-k+1), 1 \leq j^b_1 < \cdots < j^b_{r_2} \leq (n-k+1) : \forall l = 1, \ldots, r-1; j_{l+1} - j_l \geq k\}$$

and

$$J^c = \{1 \leq j^a_1 < \cdots < j^a_{r_1} \leq (n-k+1), 1 \leq j^b_1 < \cdots < j^b_{r_2} \leq (n-k+1) : \exists l = 1, \ldots, r-1; j_{l+1} - j_l < k\}.$$

Then we have the following,

Theorem 14. *Let Pois(λ) be a Poisson random variable with parameter λ. Then, as $n \to \infty$,*

$$(A_n, B_n) \xrightarrow{d} (Z_1, Z_2). \tag{14}$$

where $Z_i \in Pois\left(\frac{1}{2}\right)$ are independent random variables.

Proof. By [2, Theorem 23, Chapter 6] and Lemma 2 it suffices to show that for all $r_1, r_2 \geq 0$,

$$\sum_{J \cup J^C} \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) \to \frac{1}{r_1! r_2! \cdot 2^{r_1} 2^{r_2}} \quad \text{as } n \to \infty. \tag{15}$$

First sum over J. By Theorem 13,

$$\begin{aligned}
\sum_J \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) &= \sum_J \left(\left(\prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\prod_{l=1}^{r_2} \frac{n - j_{\beta_l}}{n^2} \right) + O\left(\frac{1}{n^{r_1+r_2+1}} \right) \right) \\
&= \sum_J \left(\prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\prod_{l=1}^{r_2} \frac{n - j_{\beta_l}}{n^2} \right) + \binom{n-k+1}{r_1+r_2} \cdot O\left(\frac{1}{n^{r_1+r_2+1}} \right)
\end{aligned}$$

Now for all $(j_1, \ldots, j_{r_1+r_2}) \in J_{r_1,n-k+2} \times J_{r_2,n-k+2} \backslash J$, there exists an l such that $j_{l+1} - j_l < k$ and therefore this set has $O\left(\binom{n-k+1}{r_1+r_2-1}\right)$ elements. Thus, the sum above can written as

$$\sum_{\substack{J_{r_1,n-k+2} \\ J_{r_2,n-k+2}}} \left(\prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\prod_{l=1}^{r_2} \frac{n - j_{\beta_l}}{n^2} \right) + O\left(\binom{n-k+1}{r_1+r_2-1} \frac{1}{n^{r_1+r_2}} \right)$$

Therefore,

$$\begin{aligned}
\sum_J \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) &= \left(\sum_{J_{r_1,n-k+2}} \prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\sum_{J_{r_2,n-k+2}} \prod_{l=1}^{r_2} \frac{n - j_{\beta_l}}{n^2} \right) + O\left(\frac{1}{n} \right) \\
&= \left(\sum_{J_{r_1,n-k+2}} \prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\sum_{k-1 \leq j_1^b < \cdots < j_{r_2}^b \leq n-1} \prod_{l=1}^{r_2} \frac{j_{\beta_l}}{n^2} \right) + O\left(\frac{1}{n} \right) \\
&= \left(\sum_{J_{r_1,n-k+2}} \prod_{l=1}^{r_1} \frac{j_{\alpha_l}}{n^2} \right) \left(\sum_{1 \leq j_1^b < \cdots < j_{r_2}^b \leq n-1} \prod_{l=1}^{r_2} \frac{j_{\beta_l}}{n^2} - \sum_{1 \leq j_1^b < \cdots < j_{r_2}^b \leq k-1} \prod_{l=1}^{r_2} \frac{j_{\beta_l}}{n^2} \right) + O\left(\frac{1}{n} \right).
\end{aligned}$$

Then by Lemma 3,

$$\begin{aligned}
\sum_J \mathbb{P}_n(x_1, \ldots, x_{j_{r_1+r_2}}) &= \left(\frac{1}{n^{2r_1}} \frac{(n-k)_{2r_1}}{2^{r_1} r_1!} \right) \left(\frac{1}{n^{2r_2}} \frac{(n)_{2r_2}}{2^{r_2} r_2!} - \frac{1}{n^{2r_2}} \frac{(k)_{2r_2}}{2^{r_2} r_1!} \right) + O\left(\frac{1}{n} \right) \\
&= \left(\frac{1}{2^{r_1} r_1!} \right) \left(\frac{1}{2^{r_2} r_2!} \right) + O\left(\frac{1}{n} \right) \\
&= \frac{1}{2^{r_1+r_2} r_1! r_2!} + O\left(\frac{1}{n} \right)
\end{aligned}$$

Now for all $(j_1, \ldots, j_{r_1+r_2}) \in J^c$, there exists an l such that $j_{l+1} - j_l < k$ and thus this set has $O\left(\binom{n-k+1}{r_1+r_2-1}\right)$ elements. Therefore, summing over J^c,

$$\sum_{J^C} \mathbb{P}_n(x_{j_1}, \ldots, x_{j_{r_1+r_2}}) = O\left(\binom{n-k+1}{r_1+r_2-1} \frac{1}{n^{r_1+r_2}} \right) = O\left(\frac{1}{n} \right).$$

By (15), combining these expressions and letting $n \to \infty$ completes the proof. □

6. Conclusion

These results show that both the distribution of a particular symbol and the joint distribution of symbols on any given diagonal are asymptotically Poisson. This verifies the conjecture given in [15] for fixed k. The diagonals are particularly interesting due to the connection between staircase tableaux and the ASEP. More specifically, the first diagonal of staircase tableaux dictates the particular state of the ASEP in which it is associated with (see [10] for more details). Of course, there are many subsequent natural questions beyond a fixed diagonal of staircase tableaux such as the asymptotic joint distribution of symbols on different diagonals. In fact, any region of staircase tableaux may be of potential interest. The approach in this paper should be adaptable to most regions, provided that restricted symbols can

be counted efficiently. Diagonals beyond a fixed k would require a different technique, and the conjecture remains open for the kth diagonal as $n, k \to \infty$.

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