The Asymptotic Distribution of Symbols on Staircase Tableaux Diagonals

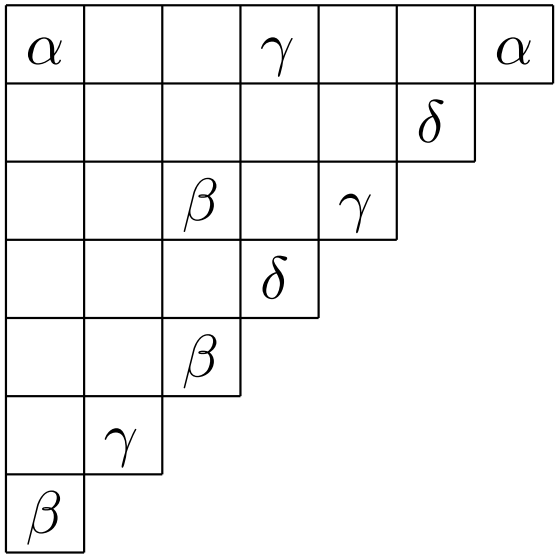
Amanda Lohss

September 15, 2016

Outline

- Definition of Staircase Tableaux
- Connections/Motivation
- Previous Results
- Results
- Open Problems

Staircase Tableaux (Corteel-Williams (2010))



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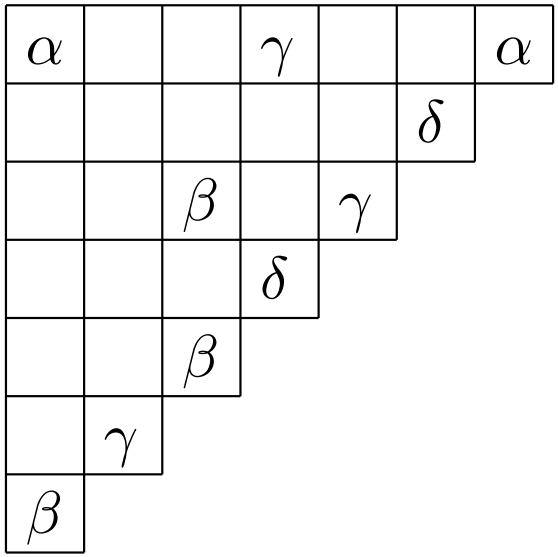

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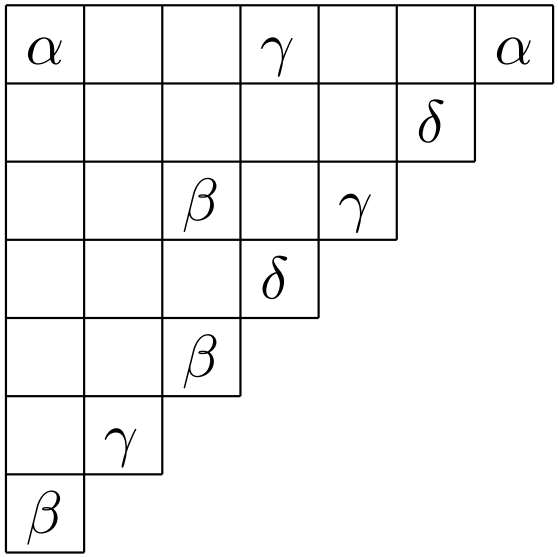

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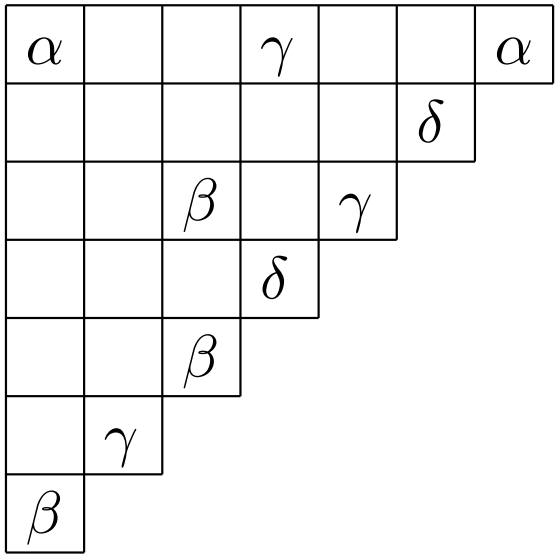

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4. All boxes in the same row and to the left of an β or δ are empty.

Preliminaries

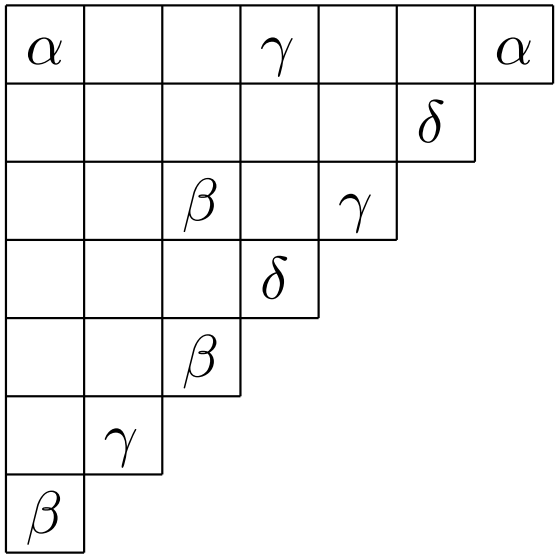

Figure: A staircase tableau with weight $\alpha^2\beta^3\gamma^3\delta^2$.

- The rows and columns in a staircase tableau are numbered from 1 through n, beginning with the box in the NW-corner and continuing south and east respectively.

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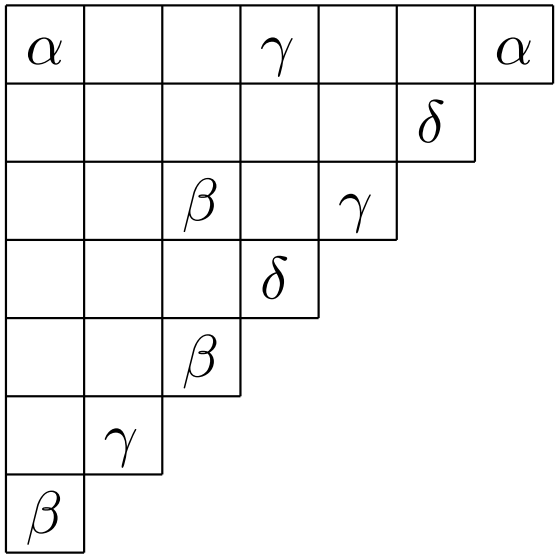

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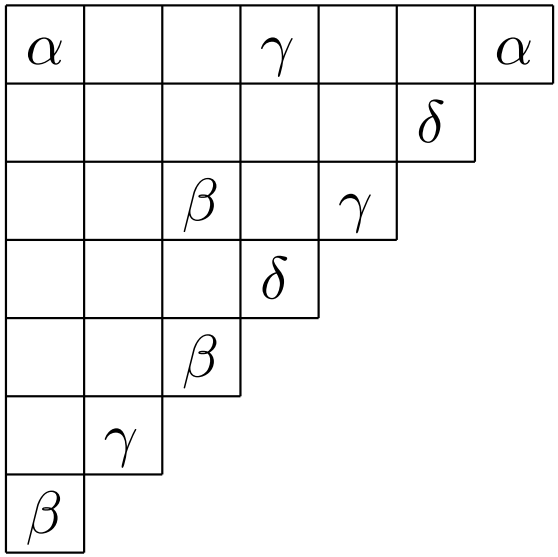

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- The weight of a staircase tableau is the product of all its symbols.

Preliminaries Cont'd

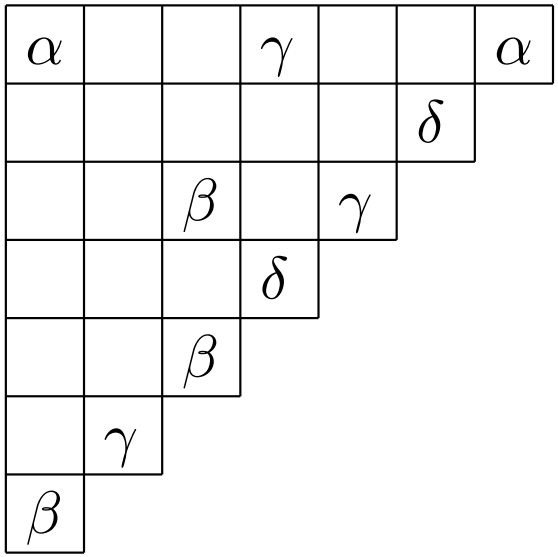

Figure: A staircase tableau with weight $\alpha^2\beta^3\gamma^3\delta^2$.

As proven by Corteel and Williams, summing over the weight of all staircase tableaux gives,

$$\sum_{S \in \mathcal{S}_n} wt(S) \quad = \quad \prod_{i=0}^{n-1} (\alpha + \beta + \delta + \gamma + i(\alpha + \gamma)(\beta + \delta)).$$

and therefore the total number of staircase tableaux is $4^n \cdot n!$.

Connections

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- Numerous other connections such as Askey-Wilson polynomials, tree–like tableaux, permutation tableaux, and permutations.

The ASEP

A Markov Chain with n sites.

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$$A\bullet\circ B \text{ to } A\circ\bullet B : \frac{u}{n+1} \qquad A\circ\bullet B \text{ to } A\bullet\circ B : \frac{q}{n+1}$$

Connection with the ASEP

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- for each α or δ on diagonal.
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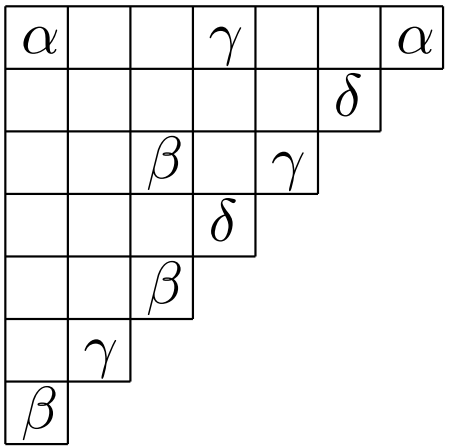

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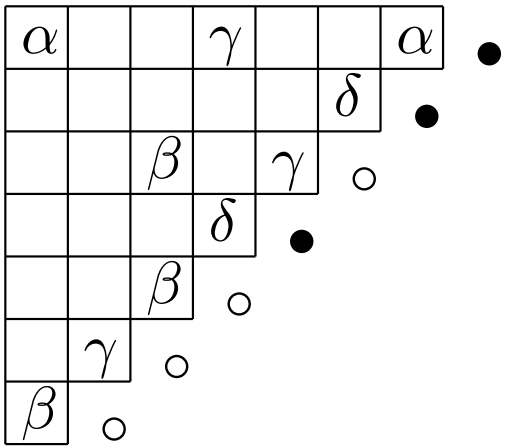

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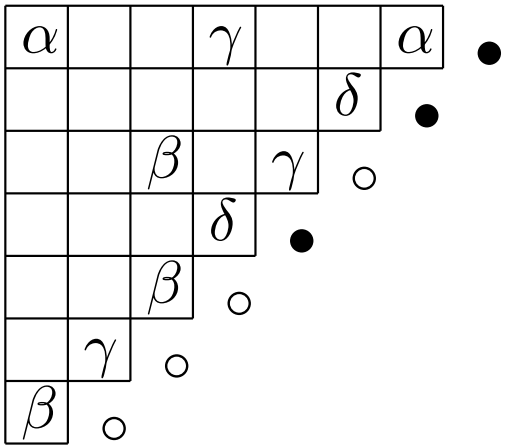

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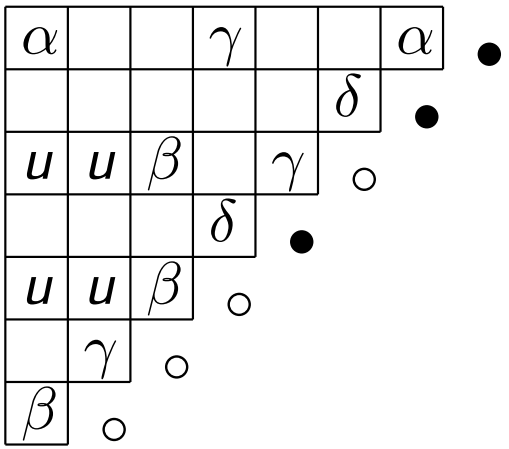

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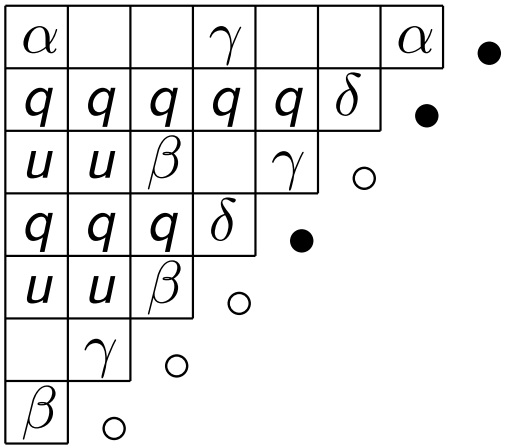

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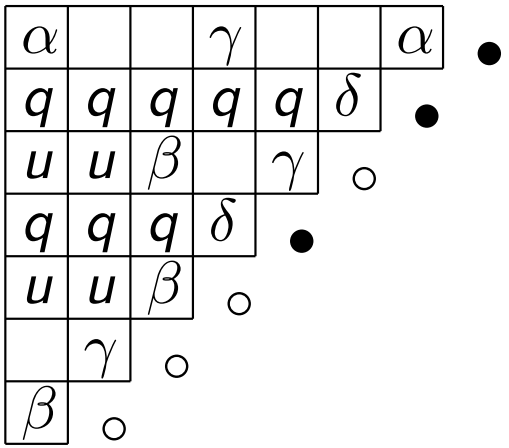

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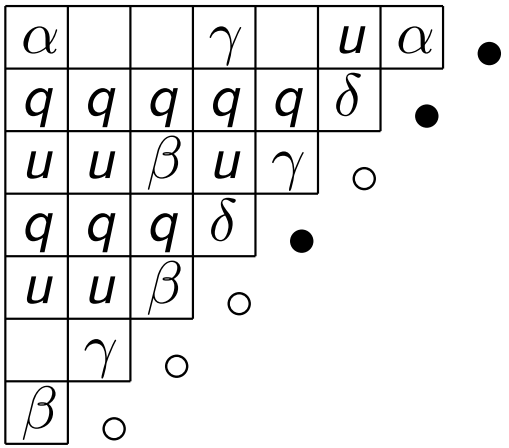

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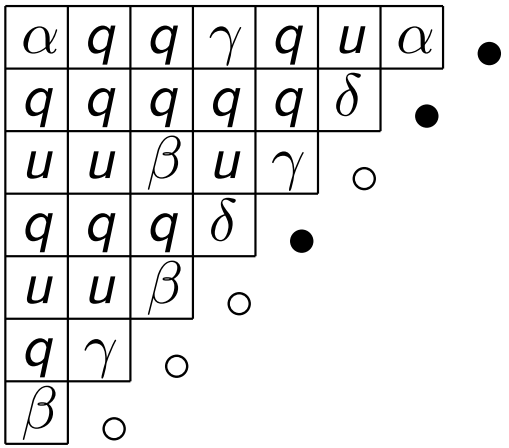

Steady State Probability of the ASEP

Theorem (Corteel and Williams)

The steady state probability that the ASEP is in state η is:

$$\frac{\sum_{T \in \mathfrak{T}} wt(T)}{\sum_{S \in \mathcal{S}_n} wt(S).}$$

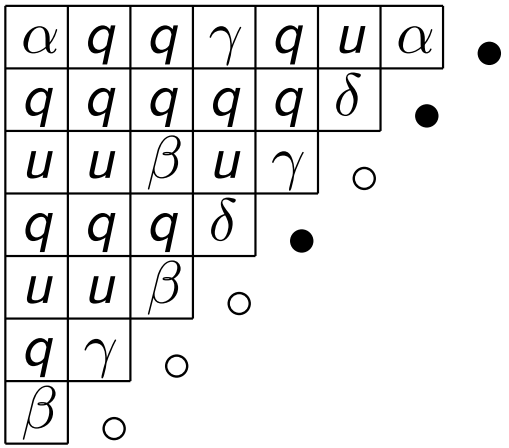

Figure: A staircase tableau and its type

● ● ○ ● ○ ○ ○

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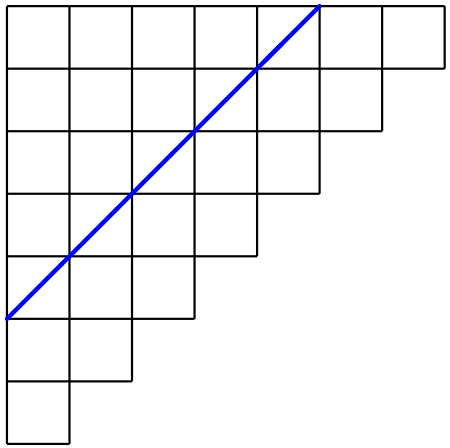
Random α/β-staircase tableaux:

$$\mathbb{P}(S_{n,\alpha,\beta} = S) = \frac{wt(S)}{\alpha^n \beta^n (a+b)^{\overline{n}}}.$$

Distribution of Symbols

Because of the ASEP, the following random variables are interesting:

1. A_n^k, the number of α's along the k^{th} diagonal.
2. B_n^k, the number of β's along the k^{th} diagonal.
3. X_n^k, the number of non-empty boxes along the k^{th} diagonal.



Some Previous Results (Hitczenko-Janson)

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Conjecture:

A_n^k and B_n^k are asymptotically Poisson ($k \geq 2$).

Results

Theorem

Let $Pois(\lambda)$ be a Poisson random variable with parameter λ. Then as $n \to \infty$:

$$A_n^k \xrightarrow{d} Pois\left(\frac{1}{2}\right) \qquad B_n^k \xrightarrow{d} Pois\left(\frac{1}{2}\right) \qquad X_n^k \xrightarrow{d} Pois\,(1)$$

Outline of Proof

Theorem (Method of Factorial Moments)

If a sequence of random variables $\{X_k\}_{k=1}^n$ *is such that*

$$\lim_{n\to\infty} \mathbb{E}(X_k)_r \to \lambda^r, \quad r = 0, 1, \ldots$$

then,

$$X_n \xrightarrow{d} Pois(\lambda)$$

For this calculation, one needs to calculate $\mathbb{P}(\alpha_{j_1} \cap \alpha_{j_2} \cdots \cap \alpha_{j_r})$.

2nd and 3rd Diagonals (with Paweł Hitczenko)

Method: conditional probability and induction

$$\mathbb{P}(\alpha_{j_1} \cap \alpha_{j_2} \cdots \cap \alpha_{j_r}) = \mathbb{P}(\alpha_{j_2} \cdots \cap \alpha_{j_r} | \alpha_{j_1}) \cdot \mathbb{P}(\alpha_{j_1}).$$

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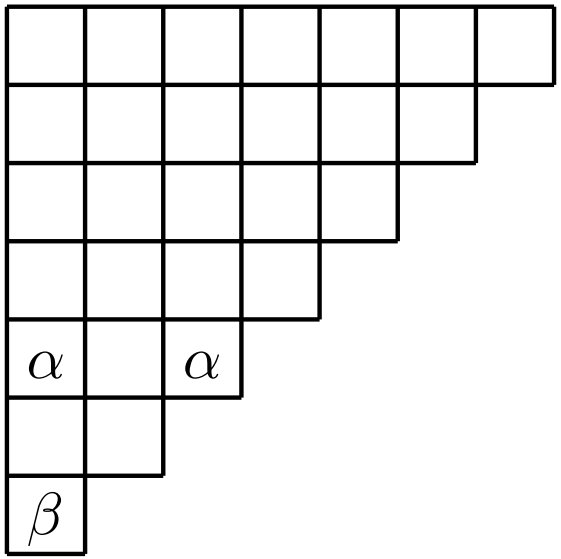

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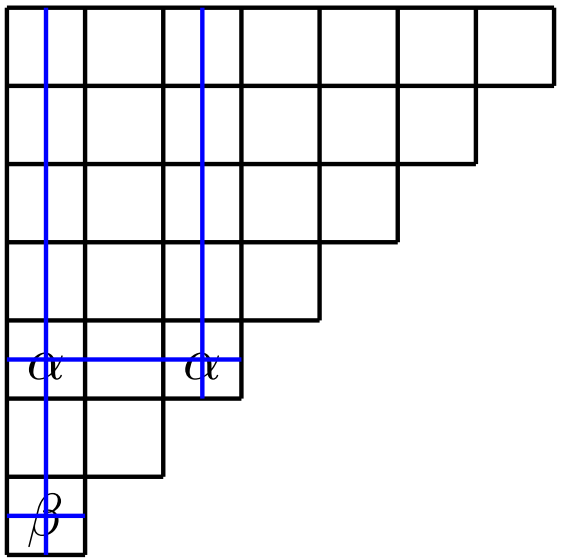

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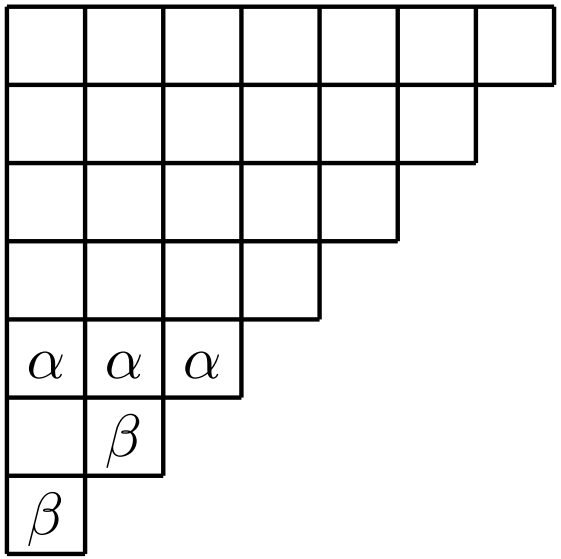

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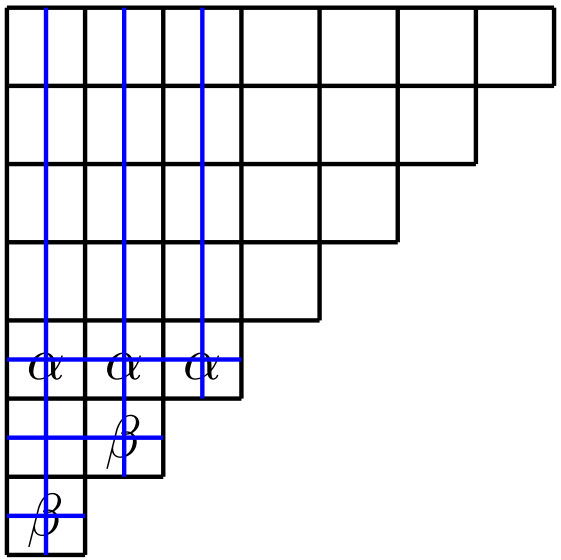

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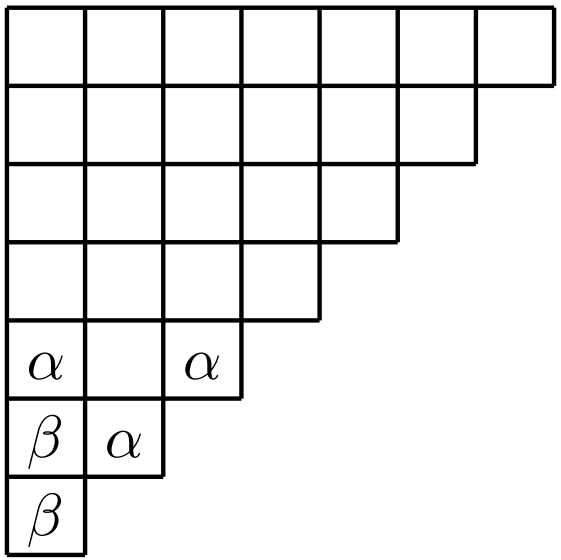

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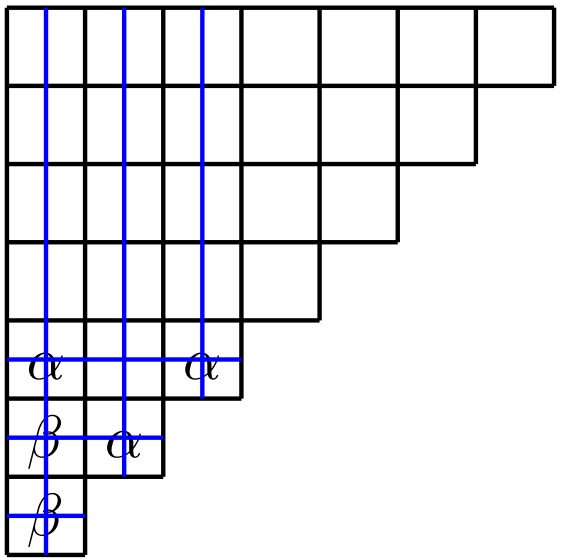

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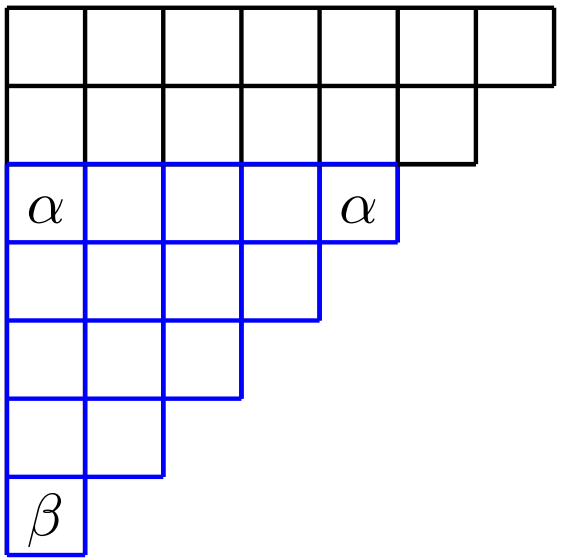

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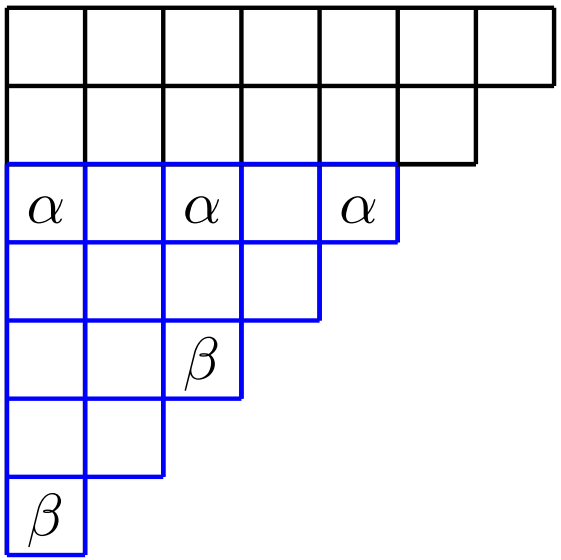

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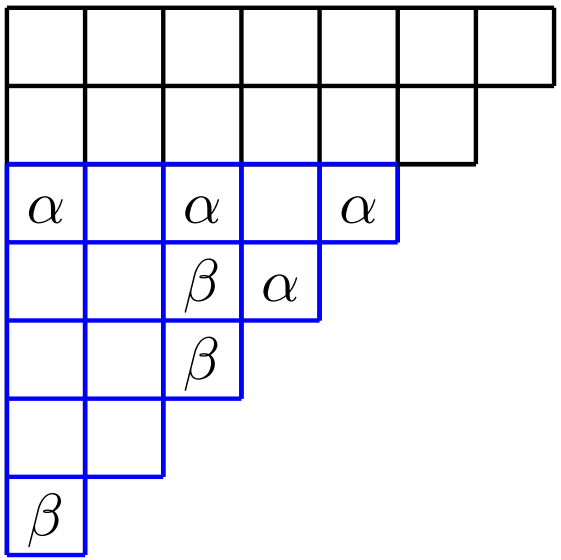
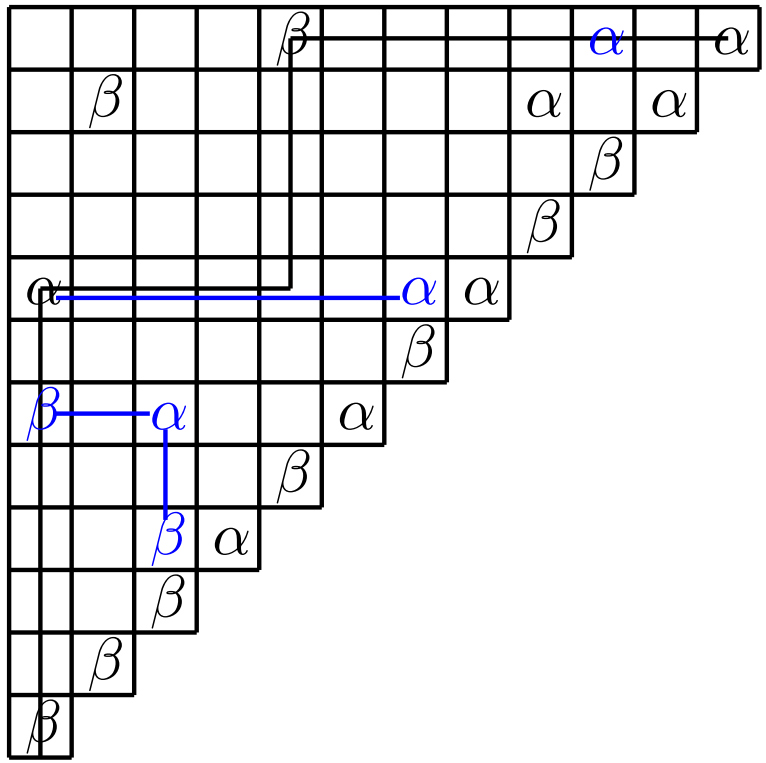


D-connected symbols

Definition

Define a symbol to be D-connected if it is not on the first diagonal and one of the following two conditions hold:

1. The symbol lies on D but is not on the kth diagonal.
2. There exists a symbol above or to the left that is D-connected or lies on D.

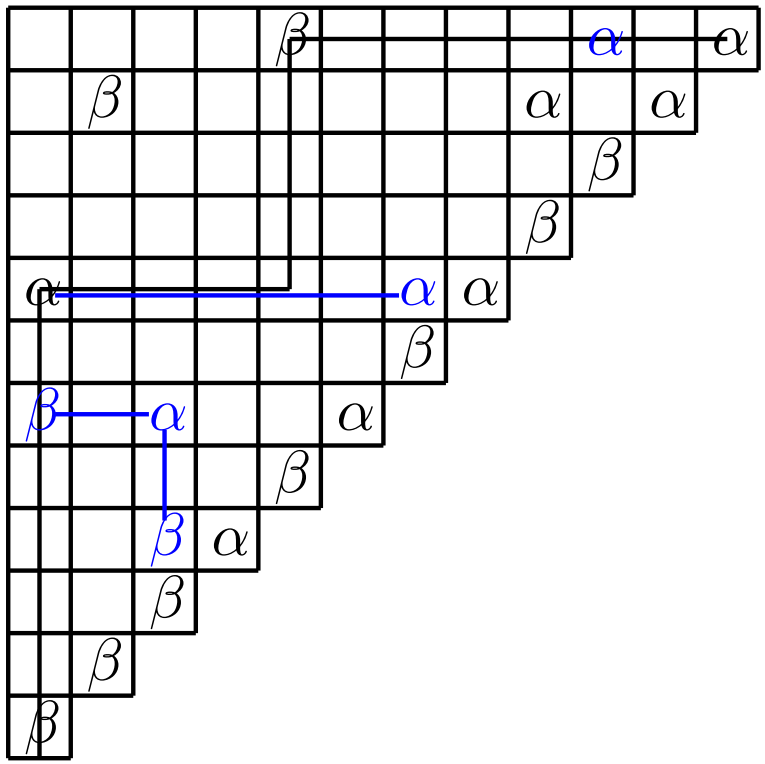


D-connected symbols

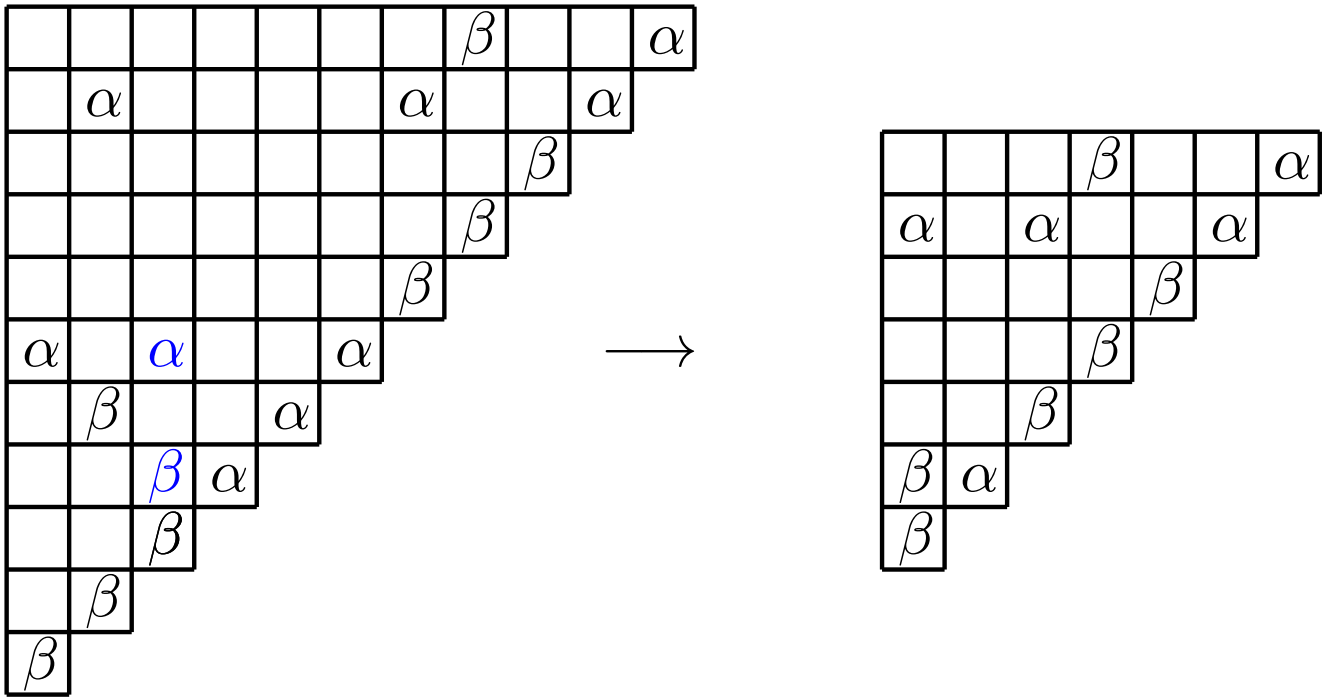
Lemma

Properties of D-connected symbols:

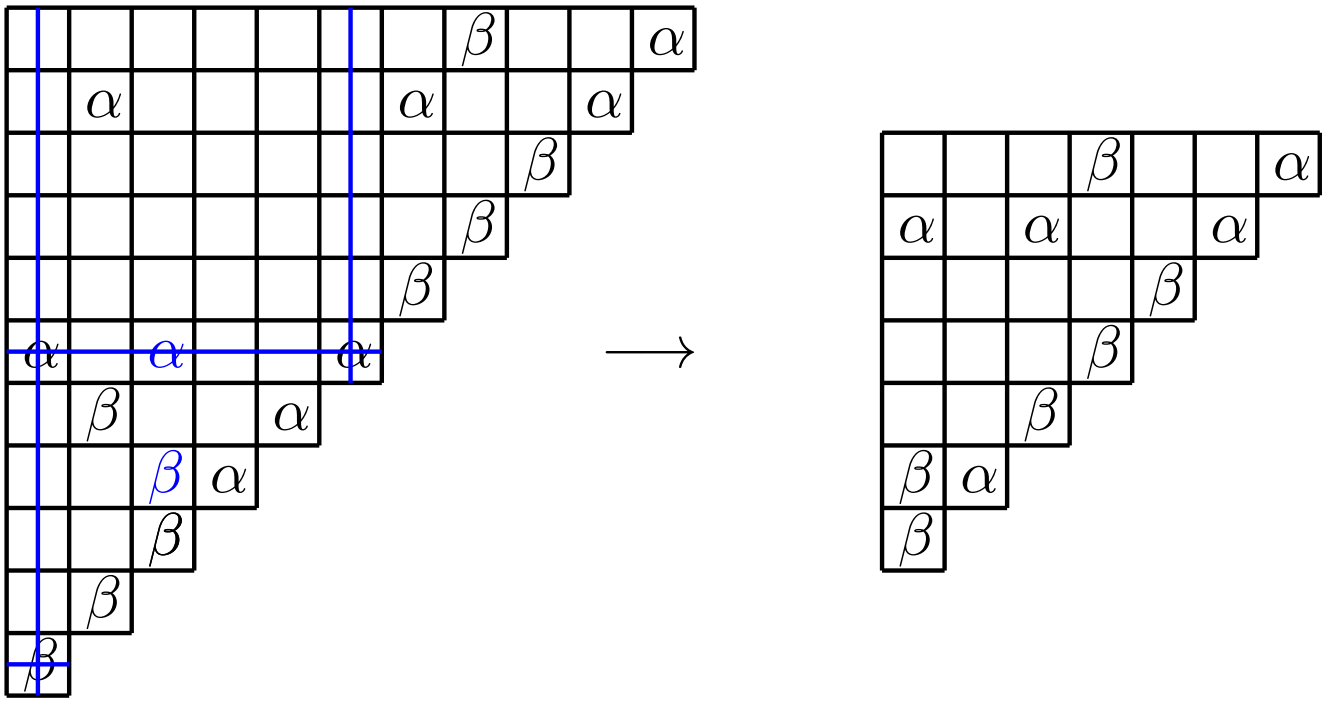
1. *Any symbol in the same column as a D-connected α or the same row as a D-connected β is also D-connected.*
2. *There are at most $k-2$ D-connected symbols.*
3. *Each D-connected symbol can be paired uniquely with an opposite symbol on the first diagonal.*



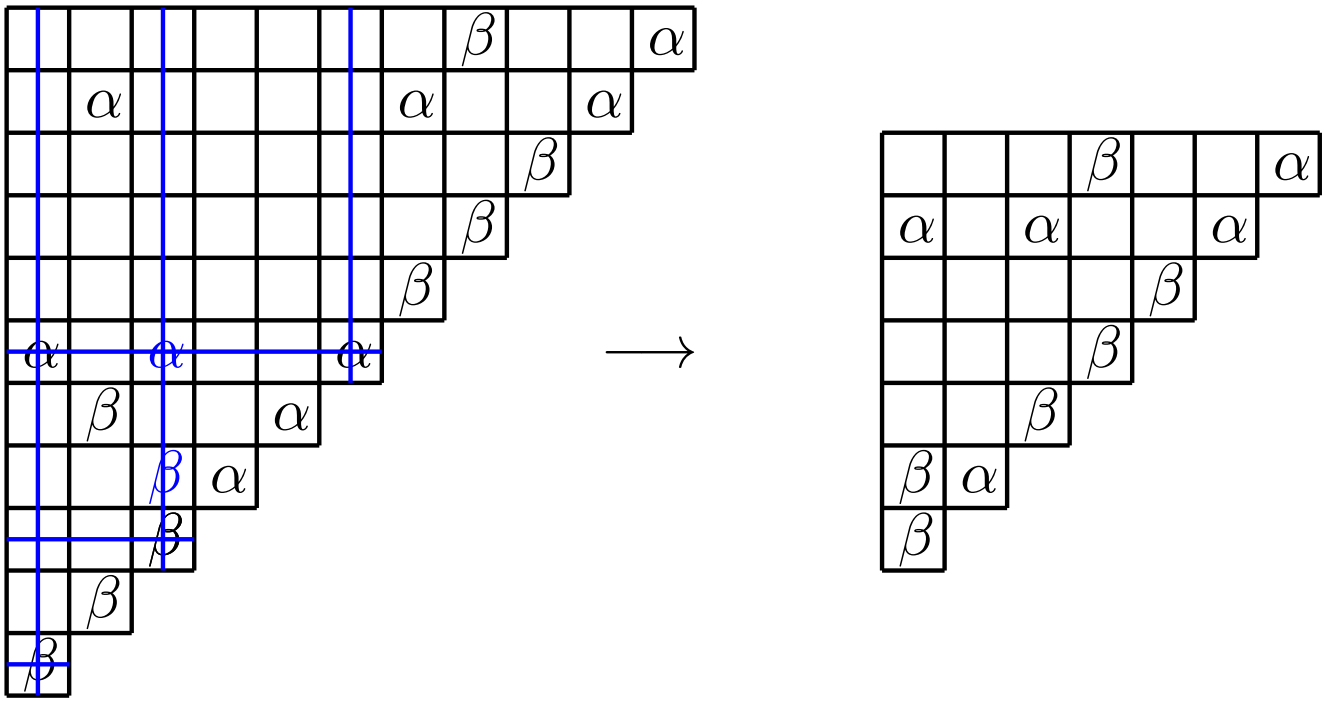
Sub-tableaux Decomposition



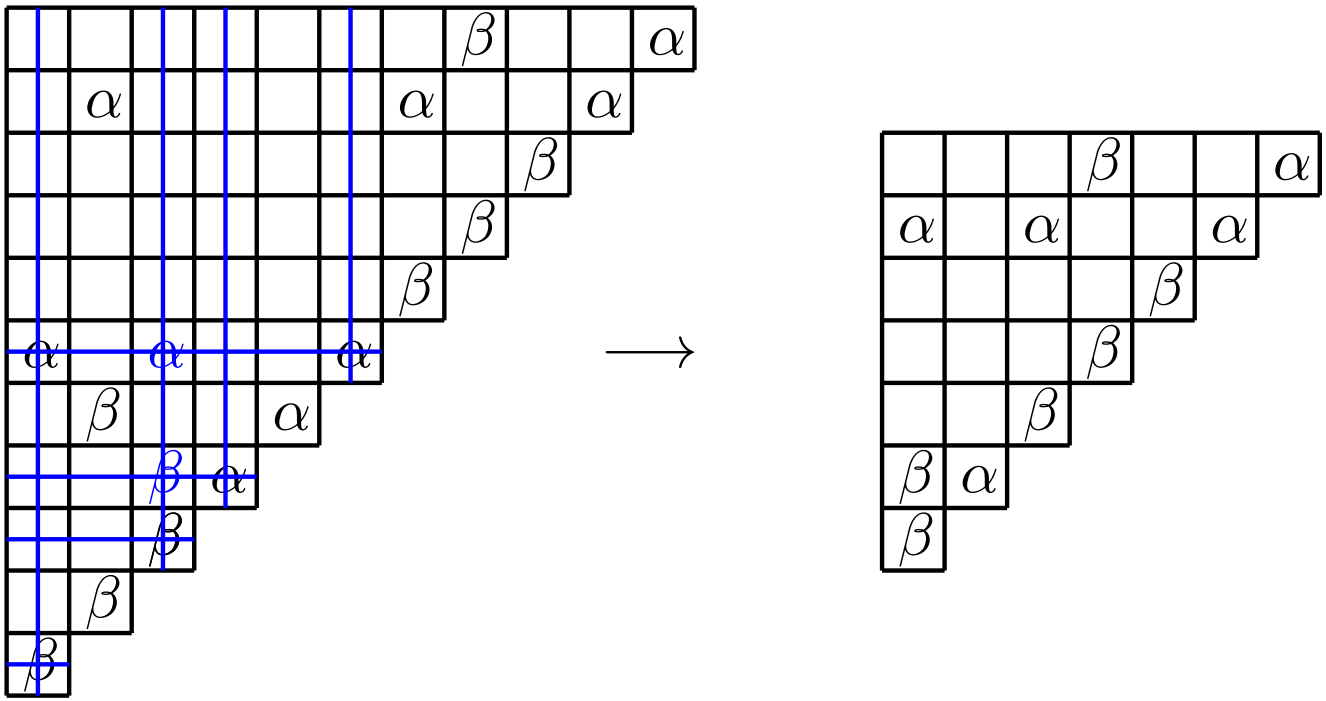
Sub-tableaux Decomposition



Sub-tableaux Decomposition



Sub-tableaux Decomposition



Notice that the weight of the larger tableau is $\alpha^2\beta\alpha^2\beta^2$ times the weight of the smaller tableau.

Arrangements of D-connected symbols

Define $\mathcal{A}_{k,h}$ to be the the set of all possible arrangements of h D-connected symbols in a tableau of size k.

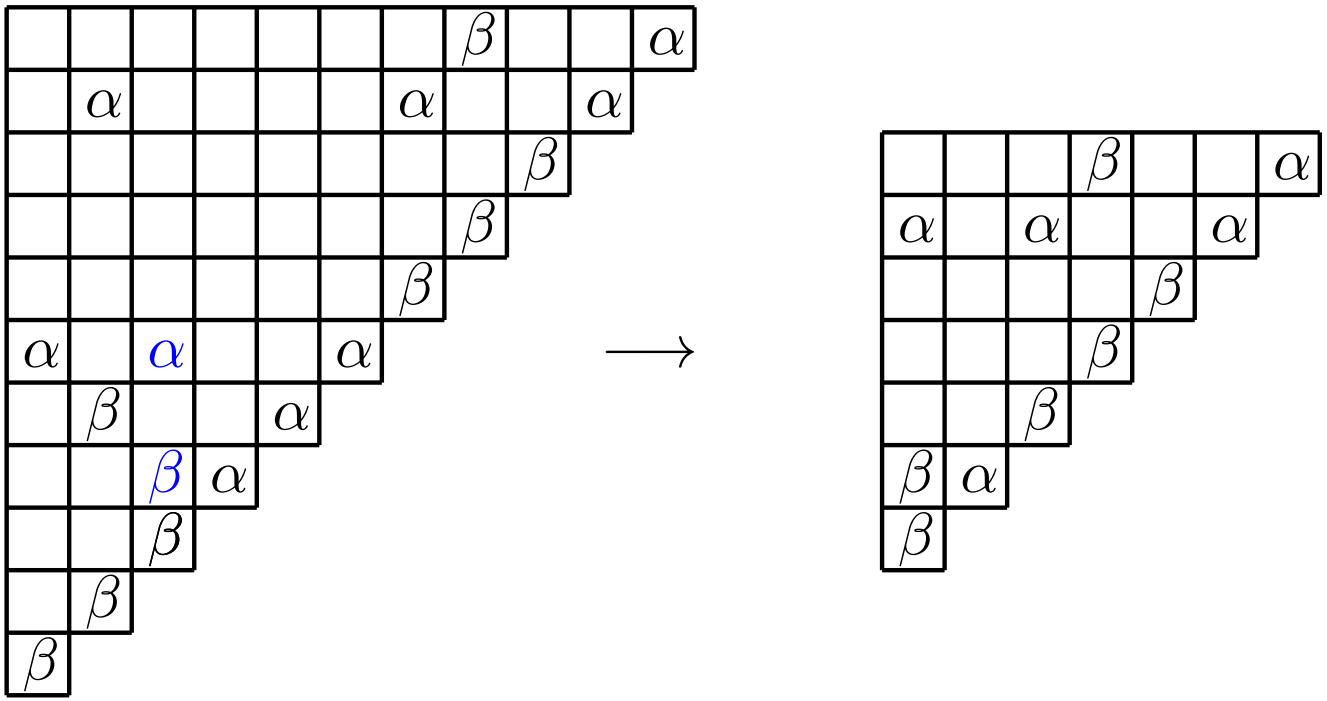
Example:

$$\mathcal{A}_{4,2} = \{\beta_1^2 \cap \beta_1^3, \beta_1^2 \cap \alpha_3^2, \beta_1^3 \cap \alpha_2^3, \beta_2^2 \cap \alpha_2^3, \beta_1^3 \cap \alpha_2^2, \alpha_2^3 \cap \alpha_3^2\}$$

Note that $\mathcal{A}_{k,0} = \{\emptyset\}$ for all $k \geq 1$.

Lemma

There exists a bijection between tableaux of size n with α_1 and triples (h, a, T) where $0 \leq h \leq k-2$, $a \in \mathcal{A}_{k,h}$ and T is a tableaux of size $n-h-2$



An example of the bijection

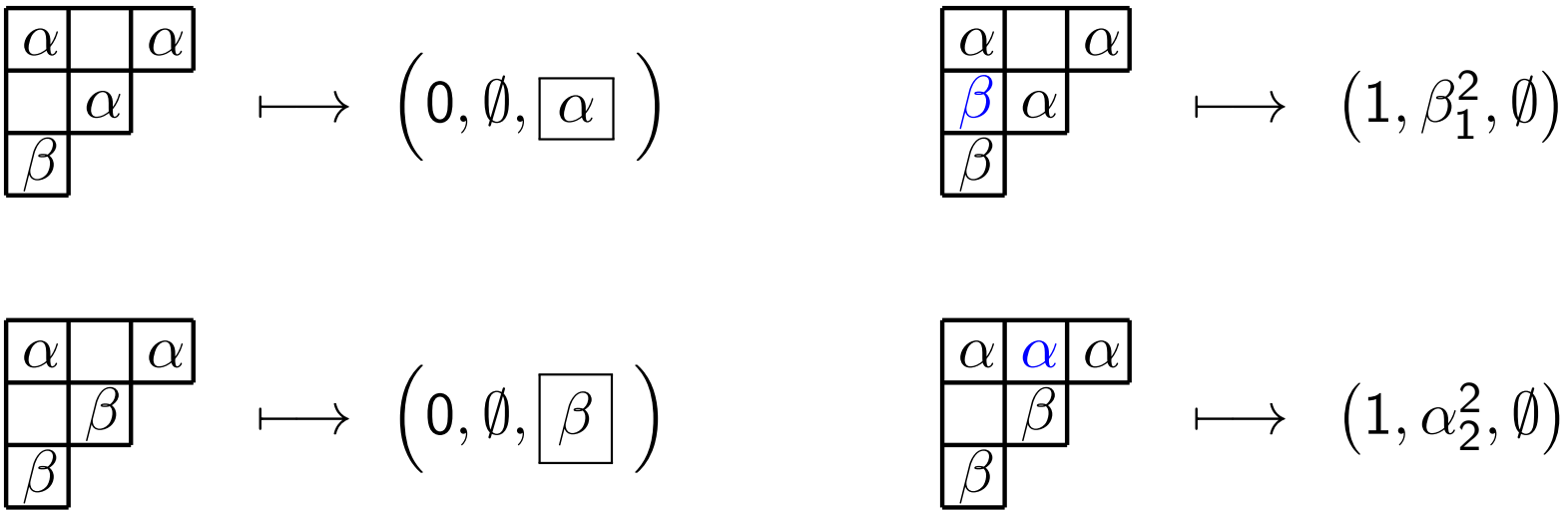
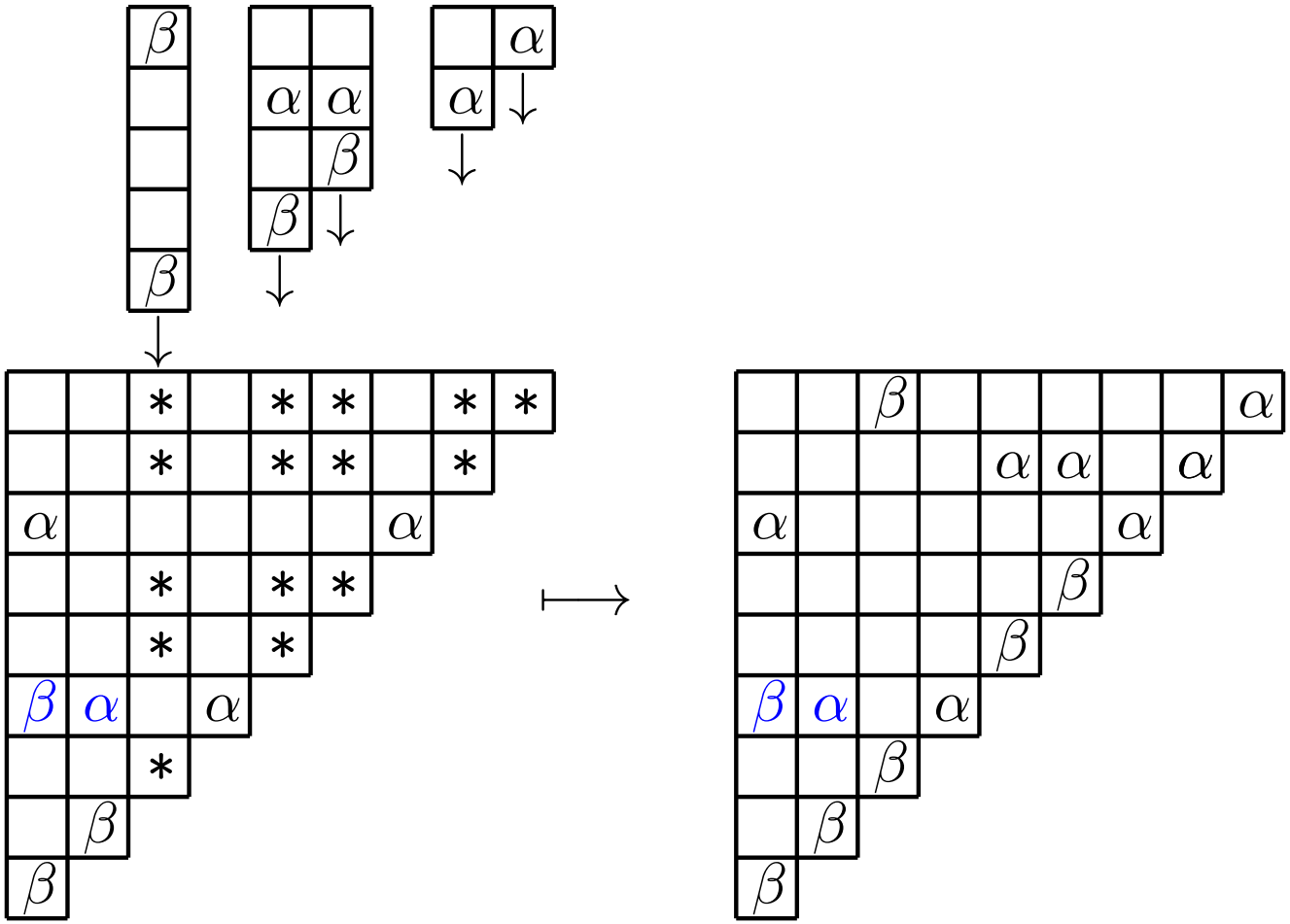


Figure: The bijection when $n = 3$ and $k = 3$.

Proof: Surjectivity



Key Result

If $C_{k,h} := |\mathcal{A}_{k,h}|$,

$$\sum_{S \in \mathcal{T}_{n,\alpha}} wt(S) = \sum_{h=0}^{k-2} C_{k,h} \alpha^{h+2} \beta^{h+1} \sum_{T \in \mathcal{T}_{n-h-2,\alpha,h}} wt(T).$$

Therefore,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_1^k, \alpha_{j_2}^k, \ldots, \alpha_{j_r}^k) =$$
$$\sum_{h=0}^{k-2} C_{k,h} \frac{b}{(n+a+b-1)_{h+2}} \mathbb{P}_{n-h-2,\alpha,\beta}(\alpha_{j_2-h-2}, \ldots, \alpha_{j_r-h-2}).$$

The Distribution of Alpha Boxes

Theorem

Let $1 \leq j_1 < ... < j_r \leq n - k + 1$. *If*

$$j_l \leq j_{l+1} - k, \quad \forall l = 1, 2, ..., r - 1$$

then

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) = \prod_{l=1}^{r} \frac{j_{r-l+1}}{n^2} + O\left(\frac{1}{n^{r+1}}\right).$$

Otherwise,

$$\mathbb{P}_{n,\alpha,\beta}(\alpha_{j_1}, ..., \alpha_{j_r}) = O\left(\frac{1}{n^r}\right).$$

Open Problems

- What about if k is not fixed?
- What about other regions?
- Asymptotic joint distribution of symbols on different diagonals?

Thank you!