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Corners in tree–like tableaux

Paweł Hitczenko[1†] and Amanda Lohss[1]

[1] *Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA*

Abstract. In this paper, we study tree–like tableaux, combinatorial objects which exhibit a natural tree structure and are connected to the partially asymmetric simple exclusion process (PASEP). There was a conjecture made on the total number of corners in tree–like tableaux and the total number of corners in symmetric tree–like tableaux. In this paper, we prove the first conjecture leaving the proof of the second conjecture to the full version of this paper. Our proofs are based on the bijection with permutation tableaux or type–B permutation tableaux and consequently, we also prove results for these tableaux.

Keywords: Tree–like tableaux, permutation tableaux, type–B permutation tableaux

1 Introduction

Tree–like tableaux are relatively new objects which were introduced in Aval et al. (2013). They are in bijection with permutation tableaux and alternative tableaux but are interesting in their own right as they exhibit a natural tree structure (see Aval et al. (2013)). They also provide another avenue in which to study the partially asymmetric simple exclusion process (PASEP), an important model from statistical mechanics. See Aval et al. (2013) and Laborde Zubieta (2015a) for more details on the connection between tree–like tableaux and the PASEP. See also Burstein (2007), Corteel and Nadeau (2009), Corteel and Williams (2007b), Corteel and Williams (2007a), Nadeau (2011), Steingrímsson and Williams (2007) and Viennot (2008) for more details on permutation and alternative tableaux.

In Laborde Zubieta (2015a), the expected number of occupied corners in tree–like tableaux and the number of occupied corners in symmetric tree–like tableaux were computed (see Section 2 for definitions). In addition, it was conjectured (see Conjectures 4.1 and 4.2 in Laborde Zubieta (2015a)) that the total number of corners in tree–like tableaux of size n is $n! \times \frac{n+4}{6}$ and the total number of corners in symmetric tree–like tableaux of size $2n+1$ is $2^n \times n! \times \frac{4n+13}{12}$.

We have proven both conjectures and in this paper, we will present the proof of the first conjecture (note that Laborde Zubieta (2015b) was able to prove the first conjecture independently using a different method). The proof of the second conjecture will be given in the full version of this paper Hitczenko and Lohss (2015). Our proofs are based on the bijection with permutation tableaux or type–B permutation tableaux and consequently, we also have results for these tableaux (see Theorems 4 and 11 below for precise statements).

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The rest of the paper is organized as follows. In the next section we introduce the necessary definitions and notation. Section 3 contains the proof of the conjecture for tree–like tableaux. Section 4 develops the tools necessary to prove the second conjecture for symmetric tree–like tableaux. The proof then follows similarly to the proof of the first conjecture and will be left to the full version of this paper Hitczenko and Lohss (2015).

2 Preliminaries

A *Ferrers diagram*, F, is a left–aligned sequence of cells with weakly decreasing rows. The *half–perimeter* of F is the number of rows plus the number of columns. The *border edges* of a Ferrers diagram are the edges of the southeast border, and the number of border edges is equal to the half–perimeter. We will occasionally refer to a border edge as a step (south or west). A *shifted Ferrers diagram* is a diagram obtained from a Ferrers diagram with k columns by adding k rows above it of lengths $k, (k-1), \ldots, 1$, respectively. The half–perimeter of the shifted Ferrers diagram is the same as the original Ferrers diagram (and similarly, the border edges are the same). The right–most cells of added rows are called *diagonal cells*.

Let us recall the following two definitions introduced in Aval et al. (2013) and Steingrímsson and Williams (2007), respectively.

Definition 1 *A tree–like tableau of size n is a Ferrers diagram of half-perimeter $n+1$ with some cells (called pointed cells) filled with a point according to the following rules:*

1. *The cell in the first column and first row is always pointed (this point is known as the root point).*
2. *Every row and every column contains at least one pointed cell.*
3. *For every pointed cell, all the cells above are empty or all the cells to the left are empty.*

Definition 2 *A permutation tableau of size n is a Ferrers diagram of half–perimeter n filled with 0's and 1's according to the following rules:*

1. *There is at least one 1 in every column.*
2. *There is no 0 with a 1 above it and a 1 to the left of it simultaneously.*

We will also need a notion of type–B tableaux originally introduced in Lam and Williams (2008). Our definition follows a more explicit description given in (Corteel and Kim, 2011, Section 4).

Definition 3 *A type–B permutation tableau of size n is a shifted Ferrers diagram of half–perimeter n filled with 0's and 1's according to the following rules:*

1. *There is at least one 1 in every column.*
2. *There is no 0 with a 1 above it and a 1 to the right of it simultaneously.*
3. *If one of the diagonal cells contains a 0 (called a diagonal 0), then all the cells in that row are 0.*

(i) (ii) (iii)

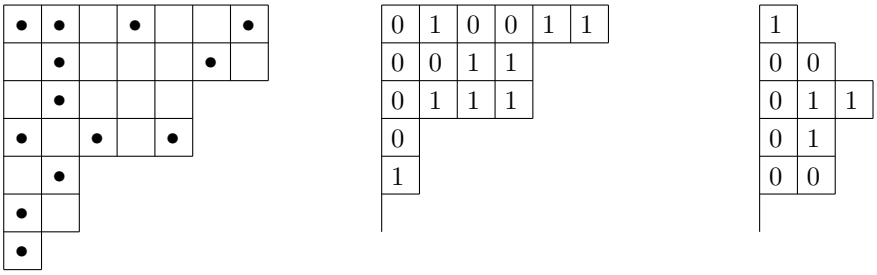

Fig. 1: (i) A tree–like tableau of size 13. (ii) A permutation tableau of size 12. (iii) A type-B permutation tableau of size 6.

Let $\mathcal{T}_n$ be the set of all tree–like tableaux of size n, $\mathcal{P}_n$ denote the set of all permutation tableaux of size n, and $\mathcal{B}_n$ denote the set of all type–B permutation tableaux of size n. In addition to these tableaux, we are also interested in *symmetric tree–like tableaux*, a subset of tree–like tableaux which are symmetric about their main diagonal (see (Aval et al., 2013, Section 2.2) for more details). As noticed in Aval et al. (2013), the size of a symmetric tree–like tableaux must be odd, and thus, we let $\mathcal{T}^{sym}_{2n+1}$ denote the set of all symmetric tree–like tableaux of size $2n+1$. It is a well–known fact that $|\mathcal{P}_n| = n!$ and $|\mathcal{B}_n| = 2^n n!$. Consequently, $|\mathcal{T}_n| = n!$ and $|\mathcal{T}^{sym}_{2n+1}| = 2^n n!$ since by Aval et al. (2013), there are bijections between these objects. We let $\mathcal{X}_n \in \{\mathcal{T}_n, \mathcal{T}^{sym}_{2n+1}, \mathcal{P}_n, \mathcal{B}_n\}$ be any of the four sets of tableaux defined above.

In permutation tableaux and type–B permutation tableaux, a *restricted* 0 is a 0 which has a 1 above it in the same column. An *unrestricted row* is a row which does not contain any restricted 0's (and for type–B permutation tableaux, also does not contain a diagonal 0). We let $U_n(T)$ denote the number of unrestricted rows in a tableau T of size n. It is also convenient to denote a topmost 1 in a column by 1_T and a right-most restricted 0 by 0_R.

Corners of a Ferrers diagram (or the associated tableau) are the cells in which both the right and bottom edges are border edges (i.e. a south step followed by a west step). In tree–like tableaux (symmetric or not) *occupied corners* are corners that contain a point.

Our proofs will rely on techniques developed in Corteel and Hitczenko (2007) (see also Hitczenko and Janson (2010)). These two papers used probabilistic language and we adopt it here, too. Thus, instead of talking about the number of corners in tableaux we let $\mathbb{P}_n$ be a probability distribution on $\mathcal{X}_n$ defined by

$$\mathbb{P}_n(T) = \frac{1}{|\mathcal{X}_n|}, \quad T \in \mathcal{X}_n, \tag{1}$$

and we consider a random variable C_n on the probability space $(\mathcal{X}_n, \mathbb{P}_n)$ defined by

$$C_n(T) = k \quad \text{if and only if } T \text{ has } k \text{ corners}, \quad T \in \mathcal{X}_n, \quad k \geq 0.$$

For convenience, let S_k indicate that the k^{th} step (border edge) is south and W_k indicate that the k^{th} step is west. Thus,

$$C_n = \sum_{k=1}^{n-1} I_{S_k, W_{k+1}}, \tag{2}$$

where I_A is the indicator random variable of the event A.

A tableau chosen from $\mathcal{X}_n$ according to the probability measure $\mathbb{P}_n$ is usually referred to as a random tableau of size n and C_n is referred to as the number of corners in a random tableau of size n. We let $\mathbb{E}_n$ denote the expected value with respect to the measure $\mathbb{P}_n$. If $c(\mathcal{X}_n)$ denotes the total number of corners in tableaux in $\mathcal{X}_n$ then, in view of (1), we have the following simple relation:

$$\mathbb{E}_n C_n = \frac{c(\mathcal{X}_n)}{|\mathcal{X}_n|} \quad \text{or, equivalently,} \quad c(\mathcal{X}_n) = |\mathcal{X}_n|\, \mathbb{E}_n C_n. \tag{3}$$

3 Corners in Tree-Like Tableaux

The main result of this section is the proof of the first conjecture of Laborde Zubieta.

Theorem 1 *(see (Laborde Zubieta, 2015a, Conjecture 4.1)) For $n \geq 2$ we have*

$$c(\mathcal{T}_n) = n! \times \frac{n+4}{6}.$$

To prove this, we will use the bijection between tree–like tableaux and permutation tableaux. According to Proposition 1.3 of Aval et al. (2013), there exists a bijection between permutation tableaux and tree–like tableaux which transforms a tree–like tableau of shape F to a permutation tableau of shape F' which is obtained from F by removing the SW–most edge from F and the cells of the left–most column (see Figure 2).

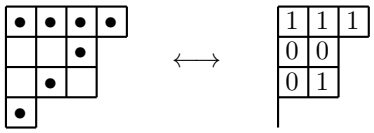

Fig. 2: An example of the bijection between permutation tableaux and tree–like tableaux of size 7.

The number of corners in F is the same as the number of corners in F' if the last edge of F' is horizontal and it is one more than the number of corners in F' if the last edge of F' is vertical. Furthermore, as is clear from a recursive construction described in (Corteel and Hitczenko, 2007, Section 2), any permutation tableau of size n whose last edge is vertical is obtained as the unique extension of a permutation tableau of size $n-1$. Therefore, there are $(n-1)!$ such tableaux and we have a simple relation

$$c(\mathcal{T}_n) = c(\mathcal{P}_n) + |\{P \in \mathcal{P}_n : \ S_n\}| = c(\mathcal{P}_n) + (n-1)!. \tag{4}$$

Thus, it suffices to determine the number of corners in permutation tableaux of size n. Since $|\mathcal{P}_n| = n!$, Equation (3) becomes

$$c(\mathcal{P}_n) = n!\, \mathbb{E}_n C_n. \tag{5}$$

In order to determine the number of corners in permutation tableaux, we first have the following result.

Theorem 2 *For permutation tableaux of size n, the probability of having a corner with border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right) = \frac{n-k+1}{n} - \frac{(n-k)^2}{n(n-1)}.$$

Proof: The theorem can be proven by using techniques developed in Corteel and Hitczenko (2007). Specifically, if $k+1 \leq n-1$ then $I_{S_k,W_{k+1}}$ is a random variable on $\mathcal{P}_{n-1}$ (denoted by $\mathcal{T}_{n-1}$ in Corteel and Hitczenko (2007) and Hitczenko and Janson (2010)). A relationship between the measures on $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$ was derived in Corteel and Hitczenko (2007) and is given by (see (Corteel and Hitczenko, 2007, Equation (7)) and (Hitczenko and Janson, 2010, Section 2, Equation (2.1))),

$$\mathbb{E}_n X_{n-1} = \frac{1}{n}\mathbb{E}_{n-1}(2^{U_{n-1}}X_{n-1}) \tag{6}$$

where X_{n-1} is any random variable defined on $\mathcal{P}_{n-1}$.

Therefore,

$$\begin{aligned}\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right) &= \mathbb{E}_n\left(I_{S_k,W_{k+1}}\right) = \frac{1}{n}\mathbb{E}_{n-1}\left(2^{U_{n-1}}I_{S_k,W_{k+1}}\right)\\ &= \frac{1}{n}\mathbb{E}_{n-1}\mathbb{E}\left(2^{U_{n-1}}I_{S_k,W_{k+1}}|\mathcal{F}_{n-2}\right),\end{aligned}$$

where $\mathcal{F}_{n-2}$ is a σ–subalgebra on $\mathcal{P}_{n-1}$ obtained by grouping into one set all tableaux in $\mathcal{P}_{n-1}$ that are obtained by extending the same tableau in $\mathcal{P}_{n-2}$ (we refer to (Hitczenko and Janson, 2010, Section 2) for a detailed explanation). Now, if $k+1 \leq n-2$ then $I_{S_k,W_{k+1}}$ is measurable with respect to the σ-algebra $\mathcal{F}_{n-2}$. Thus by the properties of conditional expectation the above is:

$$\mathbb{E}_n\left(I_{S_k,W_{k+1}}\right) = \frac{1}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\mathbb{E}\left(2^{U_{n-1}}|\mathcal{F}_{n-2}\right).$$

By (Corteel and Hitczenko, 2007, Equation (4)), the conditional distribution of U_n given U_{n-1} is given by

$$\mathcal{L}(U_n|\mathcal{F}_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}),$$

where $\mathrm{Bin}(m)$ denotes a binomial random variable with parameters m and $1/2$. By this result and the fact that $\mathbb{E}a^{\mathrm{Bin}(m)} = \left(\frac{a+1}{2}\right)^m$,

$$\begin{aligned}\frac{1}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\mathbb{E}\left(2^{U_{n-1}}|\mathcal{F}_{n-2}\right) &= \frac{1}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\mathbb{E}\left(2^{1+\mathrm{Bin}(U_{n-2})}|\mathcal{F}_{n-2}\right)\\ &= \frac{2}{n}\mathbb{E}_{n-1}I_{S_k,W_{k+1}}\left(\frac{3}{2}\right)^{U_{n-2}}\\ &= \frac{2}{n(n-1)}\mathbb{E}_{n-2}I_{S_k,W_{k+1}}3^{U_n-2}\end{aligned} \tag{7}$$

where the last step follows from (6). Iterating $(n-1)-(k+1)$ times, we obtain

$$\frac{2\cdot 3\cdot\cdots\cdot(n-k-1)}{n(n-1)\cdot\cdots\cdot(k+2)}\mathbb{E}_{k+1}I_{S_k,W_{k+1}}\,(n-k)^{U_{k+1}}\,. \tag{8}$$

Thus, we need to compute

$$\mathbb{E}_{k+1} I_{S_k, W_{k+1}} (n-k)^{U_{k+1}} \tag{9}$$

for $1 \leq k \leq n-1$ (note that $k+1 = n$ gives $\mathbb{E}_n I_{S_{n-1}, W_n}$ which is exactly the summand omitted earlier by the restriction $k+1 \leq n-1$). This can be computed as follows. First, by the tower property of the conditional expectation and the fact that S_k is $\mathcal{F}_k$–measurable, we obtain

$$\mathbb{E}_{k+1} I_{S_k, W_{k+1}} (n-k)^{U_{k+1}} = \mathbb{E}_{k+1} I_{S_k} \mathbb{E}(I_{W_{k+1}} (n-k)^{U_{k+1}} | \mathcal{F}_k).$$

And now

$$\mathbb{E}(I_{W_{k+1}} (n-k)^{U_{k+1}} | \mathcal{F}_k) = \mathbb{E}((n-k)^{U_{k+1}} | \mathcal{F}_k) - \mathbb{E}(I_{S_{k+1}} (n-k)^{U_{k+1}} | \mathcal{F}_k)$$

because the two indicators are complementary. The first conditional expectation on the right–hand side, by a computation similar to (7) (see also (Hitczenko and Janson, 2010, Equation (2.2))) is

$$(n-k)\mathbb{E}\left((n-k)^{U_{k+1}} | \mathcal{F}_k\right) = (n-k)\left(\frac{n-k+1}{2}\right)^{U_k}. \tag{10}$$

To compute the second conditional expectation, note that on the set S_{k+1}, $U_{k+1} = 1 + U_k$ so that

$$\begin{aligned} \mathbb{E}(I_{S_{k+1}} (n-k)^{U_{k+1}} | \mathcal{F}_k) &= (n-k)^{1+U_k} \mathbb{E}(I_{S_{k+1}} | \mathcal{F}_k) \\ &= (n-k)^{1+U_k} \mathbb{P}(I_{S_{k+1}} | \mathcal{F}_k) \\ &= (n-k)^{1+U_k} \frac{1}{2^{U_k}} \end{aligned}$$

where the last equation follows from the fact that for every tableau $P \in \mathcal{P}_k$ only one of its $2^{U_k(P)}$ extensions to a tableau in $\mathcal{P}_{k+1}$ has S_{k+1} (see Corteel and Hitczenko (2007); Hitczenko and Janson (2010) for more details and further explanation). Combining with (10) yields

$$\mathbb{E}(I_{W_{k+1}} (n-k)^{U_{k+1}} | \mathcal{F}_k) = (n-k)\left(\left(\frac{n-k+1}{2}\right)^{U_k} - \left(\frac{n-k}{2}\right)^{U_k}\right)$$

and thus (9) equals

$$(n-k)\mathbb{E}_{k+1}\left(I_{S_k}\left(\left(\frac{n-k+1}{2}\right)^{U_k} - \left(\frac{n-k}{2}\right)^{U_k}\right)\right).$$

The expression inside the expectation is a random variable on $\mathcal{P}_k$ so that we can use the same argument as above (based on (Corteel and Hitczenko, 2007, Equation 5) or (Hitczenko and Janson, 2010, Equation (2.1))) to reduce the size by one and obtain that the expression above is

$$\frac{n-k}{k+1}\mathbb{E}_k I_{S_k}\left((n-k+1)^{U_k} - (n-k)^{U_k}\right).$$

Furthermore, on the set S_k, $U_k = U_{k-1} + 1$ so that the above is

$$\frac{n-k}{k+1}\mathbb{E}_k\left(\left((n-k+1)^{1+U_{k-1}} - (n-k)^{1+U_{k-1}}\right)\mathbb{E}(I_{S_k} | \mathcal{F}_{k-1})\right),$$

which, by the same argument as above, equals

$$\frac{n-k}{k+1}\mathbb{E}_k\left(\left((n-k+1)^{1+U_{k-1}}-(n-k)^{1+U_{k-1}}\right)\frac{1}{2^{U_{k-1}}}\right).$$

After reducing the size one more time we obtain

$$\frac{n-k}{(k+1)k}\left(\mathbb{E}_{k-1}\,(n-k+1)^{1+U_{k-1}}-\mathbb{E}_{k-1}\,(n-k)^{1+U_{k-1}}\right). \tag{11}$$

As computed in (Hitczenko and Janson, 2010, Equation (2.4)) for a positive integer m the generating function of U_m is given by

$$\mathbb{E}_m z^{U_m}=\frac{\Gamma(z+m)}{\Gamma(z)m!}.$$

(There is an obvious omission in (2.4) there; the $z+n$ in the third expression should be $z+n-1$.) Using this with $m=k-1$ and $z=n-k+1$ and then with $z=n-k$ we obtain

$$\mathbb{E}_{k-1}\left((n-k+1)^{1+U_{k-1}}\right)=(n-k+1)\frac{(n-1)!}{(n-k)!(k-1)!} \tag{12}$$

and

$$\mathbb{E}_{k-1}\left((n-k)^{1+U_{k-1}}\right)=(n-k)\frac{(n-2)!}{(n-k-1)!(k-1)!}. \tag{13}$$

Combining Equations (8), (11), (12), and (13),

$$\begin{aligned}&\mathbb{E}_n\left(I_{S_k,W_{k+1}}\right)=\\&\quad\frac{(n-k-1)!(k+1)!}{n!}\cdot\frac{n-k}{k(k+1)}\left(\frac{(n-k+1)(n-1)!}{(k-1)!(n-k)!}-\frac{(n-k)(n-2)!}{(k-1)!(n-k-1)!}\right)\\&\quad=\frac{n-k+1}{n}-\frac{(n-k)^2}{n(n-1)},\end{aligned}$$

and the conclusion follows. □

The relationship between permutation tableaux and tree–like tableaux given by (4) allows us to deduce the following corollary to Theorem 6.

Corollary 3 *For tree–like tableaux of size n, $n\geq 2$, the probability of having a corner with border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{S_k,W_{k+1}}\right)=\begin{cases}\frac{n-k+1}{n}-\frac{(n-k)^2}{n(n-1)} & k=1,\ldots,n-1;\\ \frac{1}{n} & k=n.\end{cases}$$

Finally, we establish the following result which, when combined with (4) and (5), completes the proof of Theorem 1.

Theorem 4 *For permutation tableaux of size n we have*

$$\mathbb{E}_n C_n=\frac{n+4}{6}-\frac{1}{n}.$$

Proof: In view of (2) we are interested in

$$\mathbb{E}_n\left(\sum_{k=1}^{n-1} I_{S_k,W_{k+1}}\right) = \sum_{k=1}^{n-1} \mathbb{E}_n\left(I_{S_k,W_{k+1}}\right).$$

Therefore, the result is obtained by summing the expression from Theorem 2 from $k = 1$ to $n - 1$. □

To conclude this section, note that Theorem 1 could also be obtained by summing the expression from Corollary 3 from $k = 1$ to n.

4 Corners in Symmetric Tree-Like Tableaux

The main result of this section concerns the second conjecture of Laborde Zubieta.

Theorem 5 *(see (Laborde Zubieta, 2015a, Conjecture 4.2)) For $n \geq 2$ we have*

$$c(\mathcal{T}_{2n+1}^{sym}) = 2^n \times n! \times \frac{4n+13}{12}.$$

As in Section 3, we will use a bijection between symmetric tree–like tableaux and type–B permutation tableaux to relate the corners of $\mathcal{T}_{2n+1}^{sym}$ to the corners of $\mathcal{B}_n$. In Section 2.2 of Aval et al. (2013), it was mentioned that there exists such a bijection; however, no details were given. Thus, we give a description of one such bijection which will be useful to us (see Figure 3).

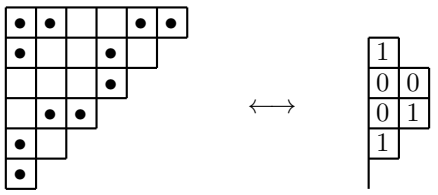

Fig. 3: An example of the bijection Fas defined in Lemma 6 between type–B permutation tableaux of size 5 and symmetric tree–like tableaux of size 11.

Lemma 6 *Consider $F : \mathcal{T}_{2n+1}^{sym} \to \mathcal{B}_n$ defined by the following rules,*

1. *Replace the topmost point in each column with 1_T's.*
2. *Replace the leftmost points in each row with 0_R's*
3. *Fill in the remaining cells according to the rules of type–B permutation tableaux.*
4. *Remove the cells above the diagonal.*
5. *Remove the first column.*

and $F^{-1} : \mathcal{B}_n \to \mathcal{T}_{2n+1}^{sym}$ defined by:

1. *Add a column and point all cells except those in a restricted row.*
2. *Replace all 0_R's with points unless that 0_R is in the same row as a diagonal 0.*
3. *Replace all non-diagonal 1_T's with points.*
4. *Delete the remaining numbers, add a pointed box in the upper–left–hand corner (the root point), and then add the boxes necessary to make the tableau symmetric.*

Then F is a bijection between $\mathcal{T}_{2n+1}^{sym}$ and $\mathcal{B}_n$.

Proof: The details of this proof are straightforward and will be given in the full version of this paper Hitczenko and Lohss (2015). □

As mentioned earlier, Lemma 6 will allow us to relate the corners of symmetric tree–like tableaux to the corners of type–B permutation tableaux. To carry out the calculations for type–B permutation tableaux we will develop techniques similar to those developed in Corteel and Hitczenko (2007) for permutation tableaux. First, we briefly describe an extension procedure for B–type tableaux that mimics a construction given in (Corteel and Hitczenko, 2007, Section 2). Fix any $B \in \mathcal{B}_{n-1}$ and let $U_{n-1} = U_{n-1}(B)$ be the number of unrestricted rows in B. We can extend the size of B to n by inserting a new row or a new column. The details of this insertion will be left for the full version of this paper. However, if U_n is the number of unrestricted rows in the extended tableaux, $U_n = 1, \ldots, U_{n-1}+1$, the (conditional) probability that $U_n = U_{n-1} + 1$ is given by inserting a row,

$$\mathbb{P}(U_n = U_{n-1} + 1 | \mathcal{F}_{n-1}) = \mathbb{P}(S_n | \mathcal{F}_{n-1}) = \frac{1}{2^{U_{n-1}+1}}. \tag{14}$$

(Here, analogously to permutation tableaux (see the proof of Theorem 4 above or (Hitczenko and Janson, 2010, Section 2)) $\mathcal{F}_{n-1}$ is a σ–subalgebra on $\mathcal{B}_n$ obtained by grouping together all tableaux in $\mathcal{B}_n$ that are obtained as the extension of the same tableau from $\mathcal{B}_{n-1}$.) The (conditional) probability of the remaining cases is given by inserting a column,

$$\mathbb{P}(U_n = k | \mathcal{F}_{n-1}) = \frac{1}{2^{U_{n-1}+1}} \left(\binom{U_{n-1}}{k-1} + \binom{U_{n-1}}{k-1} \right) = \frac{1}{2^{U_{n-1}}} \binom{U_{n-1}}{k-1},$$

for $k = 1, \ldots, U_{n-1}$. This agrees with (14) when $k = U_{n+1}$. Thus,

$$\mathcal{L}(U_n | \mathcal{F}_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}),$$

where the left–hand side means the conditional distribution of U_n given U_{n-1} and $\mathrm{Bin}(m)$ denotes a binomial random variable with parameters m and $1/2$. Note that this is the same relationship as for permutation tableaux (see (Hitczenko and Janson, 2010, Equation (2.2)) or (Corteel and Hitczenko, 2007, Equation 4)).

As in the case of permutation tableaux, the uniform measure $\mathbb{P}_n$ on $\mathcal{B}_n$ induces a measure (still denoted by $\mathbb{P}_n$) on $\mathcal{B}_{n-1}$ via a mapping $\mathcal{B}_n \to \mathcal{B}_{n-1}$ that assigns to any $B' \in \mathcal{B}_n$ the unique tableau of size $n-1$ whose extension is B'. These two measures on $\mathcal{B}_{n-1}$ are not identical, but the relationship between them

can be easily calculated (see (Corteel and Hitczenko, 2007, Section 2) or (Hitczenko and Janson, 2010, Section 2) for more details and calculations for permutation tableaux). Namely,

$$\mathbb{P}_n(B) \quad = \quad 2^{U_{n-1}(B)+1}\frac{|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|}\mathbb{P}_{n-1}(B), \quad B \in \mathcal{B}_{n-1}.$$

This relationship implies that for any random variable X on $\mathcal{B}_{n-1}$,

$$\mathbb{E}_n X \quad = \quad \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|}\mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})}X). \tag{15}$$

This allows us to provide a direct proof of the following well known fact,

Proposition 7 *For all $n \geq 0$, $|\mathcal{B}_n| = 2^n n!$.*

Proof: By considering all the extensions of a type–B permutation tableau of size $n-1$, we have the following relationship,

$$|\mathcal{B}_n| = \sum_{B \in \mathcal{B}_{n-1}} 2^{U_{n-1}(B)+1}.$$

Thus,

$$\begin{aligned}
|\mathcal{B}_n| &= |\mathcal{B}_{n-1}|\mathbb{E}_{n-1}\left(2^{U_{n-1}+1}\right) \\
&= 2|\mathcal{B}_{n-1}|\mathbb{E}_{n-1}\mathbb{E}\left(2^{1+\mathrm{Bin}(U_{n-2})}|U_{n-2}\right) \\
&= 2\cdot 2|\mathcal{B}_{n-1}|\mathbb{E}_{n-1}\left(\frac{3}{2}\right)^{U_{n-2}} \\
&= 2\cdot 2|\mathcal{B}_{n-1}|\frac{2|\mathcal{B}_{n-2}|}{|\mathcal{B}_{n-1}|}\mathbb{E}_{n-2}\left(2^{U_{n-2}}\left(\frac{3}{2}\right)^{U_{n-2}}\right) \\
&= 2^2\cdot 2!\,|\mathcal{B}_{n-2}|\mathbb{E}_{n-2}3^{U_{n-2}}.
\end{aligned}$$

Iterating n times,

$$\begin{aligned}
|\mathcal{B}_n| &= 2^3\cdot 3!\,|\mathcal{B}_{n-3}|\mathbb{E}_{n-3}4^{U_{n-3}} = 2^{n-1}(n-1)!|\mathcal{B}_1|\mathbb{E}_1 n^{U_1} \\
&= 2^n n!,
\end{aligned}$$

where the final equality holds because $|\mathcal{B}_1| = 2$ and $U_1 \equiv 1$. □

Given Proposition 7, (15) reads

$$\mathbb{E}_n X = \frac{1}{n}\mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})}X). \tag{16}$$

This is exactly the same expression as (Corteel and Hitczenko, 2007, Equation (7)) which means that the relationship between $\mathbb{E}_n$ and $\mathbb{E}_{n-1}$ is the same regardless of whether we are considering $\mathcal{P}_n$ or $\mathcal{B}_n$. Thus, any computation for B–type tableaux based on (16) will lead to the same expression as the analogous computation for permutation tableaux based on (Corteel and Hitczenko, 2007, Equation (7)).

Now we have the tools necessary to obtain a relationship between corners in symmetric tree–like tableaux and type–B permutation tableaux which is analogous to (4).

Lemma 8 *The number of corners in symmetric tree–like tableaux is given by,*

$$c(\mathcal{T}_{2n+1}^{sym}) \quad = \quad 2c(\mathcal{B}_n) + 2^n(n-1)! + 2^{n-1}n!. \tag{17}$$

Proof: The bijection described in Lemma 6 leads to the following relationship,

$$c(\mathcal{T}_{2n+1}^{sym}) = 2c(\mathcal{B}_n) + 2|\{B \in \mathcal{B}_n : \ S_n\}| + |\{B \in \mathcal{B}_n : \ W_1\}|. \tag{18}$$

The result is then obtained by the extension process described above. The details will be given in the full version of this paper Hitczenko and Lohss (2015). □

It follows from Lemma 8 that to prove Theorem 5, it suffices to determine the number of corners in type–B permutation tableaux of size n. Since $|\mathcal{B}_n| = 2^n n!$, Equation (3) becomes

$$c(\mathcal{B}_n) = 2^n n!\, \mathbb{E}_n C_n. \tag{19}$$

In order to determine the number of corners in type–B permutation tableaux, we first have the following result.

Theorem 9 *For type–B permutation tableaux of size n, the probability of having a corner with border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{M_k=S, M_{k+1}=W}\right) \quad = \quad \frac{n-k+1}{2n} - \frac{(n-k)^2}{4n(n-1)}.$$

Proof: The proof is similar to the proof of Theorem 2, using the techniques developed in this section for type–B permutation tableaux. The details will be given in the full version of this paper Hitczenko and Lohss (2015). □

The relationship between permutation tableaux and tree–like tableaux given by (17) allows us to deduce the following corollary to Theorem 9.

Corollary 10 *For symmetric tree–like tableaux of size $2n+1$, $n \geq 2$, the probability of having a corner with border edges k and $k+1$ is given by*

$$\mathbb{P}_n\left(I_{S_k, W_{k+1}}\right) = \begin{cases} \frac{1}{2n} & k = 1 \\ \frac{k}{2n} - \frac{(k-1)^2}{4n(n-1)} & k = 2, \ldots n, \\ \frac{1}{2} & k = n+1 \\ \frac{2n-k+2}{2n} - \frac{(2n-k+1)^2}{4n(n-1)} & k = n+2, \ldots 2n \\ \frac{1}{2n} & k = 2n+1. \end{cases}$$

Finally, we establish the following result which, when combined with (17) and (19), completes the proof of Theorem 5.

Theorem 11 *For type–B permutation tableaux of size n we have*

$$\mathbb{E}_n C_n = \frac{4n+7}{24} - \frac{1}{2n}.$$

Proof: The result is obtained by summing the expression from Theorem 9 from $k = 1$ to $n-1$. □

To conclude this section, note that Theorem 5 could also be obtained by summing the expression from Corollary 10 from $k = 1$ to $2n+1$.

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