Corners in Tree–Like Tableaux

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- Tree–Like Tableaux
- Corners in Tree–Like Tableaux
- Occupied Corners in Tree–Like Tableaux
- Diagonal Boxes in Symmetric Tree–Like Tableaux

Tree–Like Tableaux (Aval, Boussicault, Nadeau 2011)

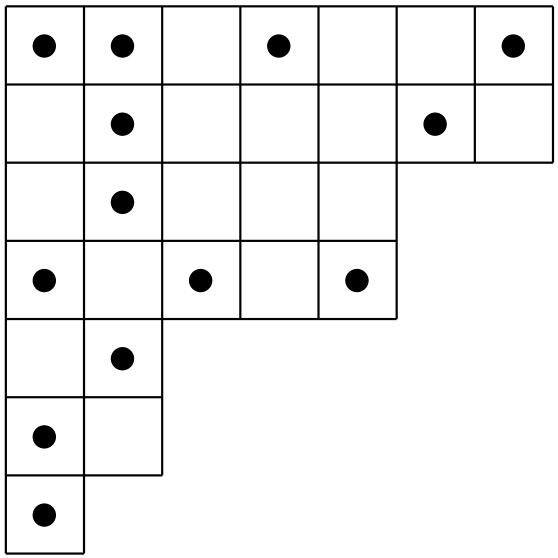


Figure: A tree–like tableaux of size 13. There are $n!$ tableaux of size n.

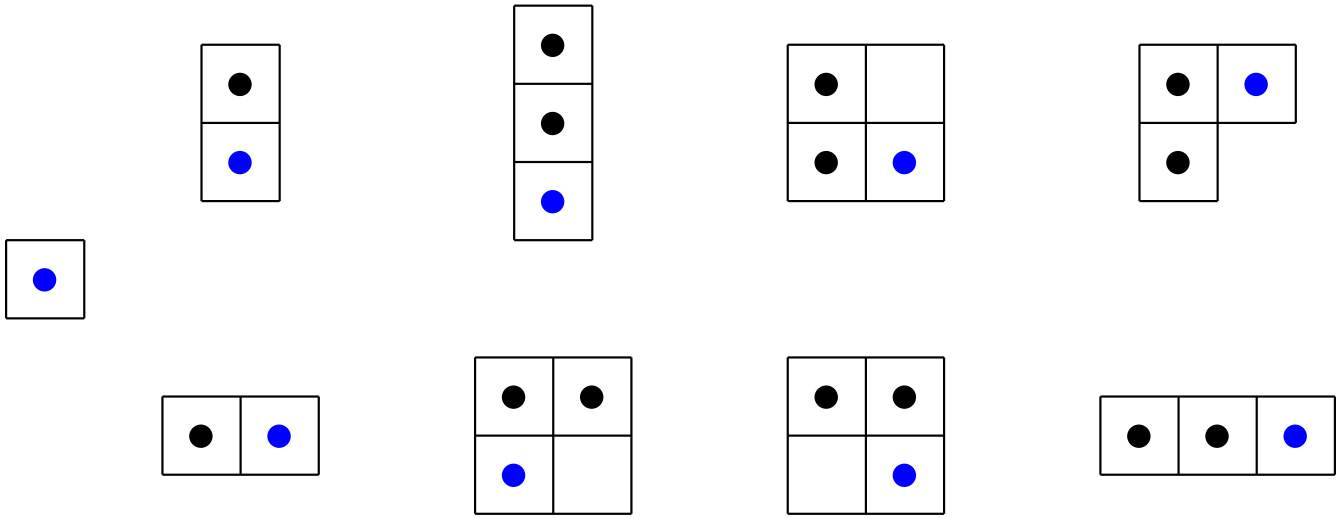
Definition

A tree-like tableaux of size n is a Ferrers diagrams of half-perimeter $n+1$ such that,

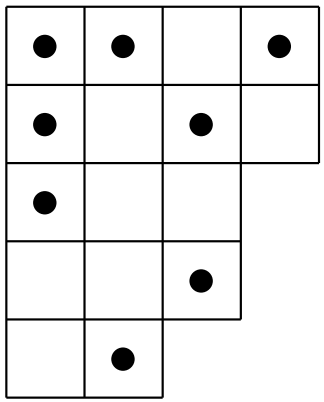
1. The box in the first column and first row is pointed.
2. Either all boxes to the left of a pointed box is empty or all boxes above are empty.
3. Every row and every column contains at least one point.

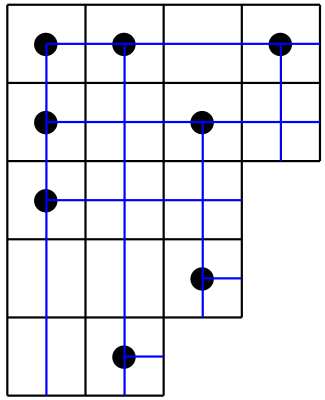
Insertion Procedure

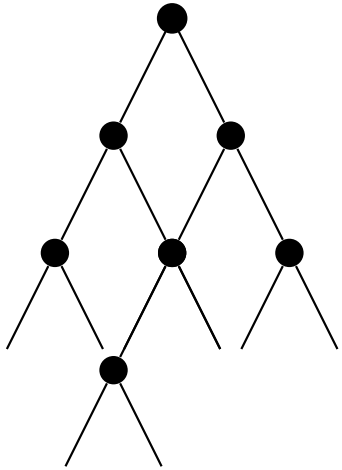
- Definition: The "special point" is the right-most point among those that occur at the bottom of a column.
- Add a pointed column for each north step and a pointed row for each west step.
- If the step is below the special point, add a ribbon.



Natural Tree Structure







There is a bijection between tree–like tableaux and permutations which “preserves trees”.

The PASEP

Tree-like tableaux provide a combinatorial formula for the partially asymmetric simple exclusion process (PASEP), an important particle model with applications in physics, biology and biochemistry.

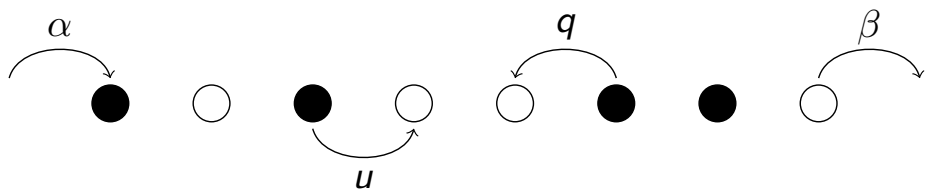


Figure: An example of the PASEP as defined by a Markov chain of size 8.

Connection with the PASEP

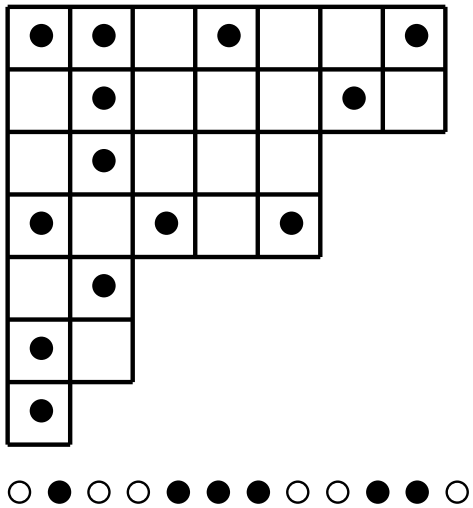


Figure: A tree–like tableau and its associated state of the PASEP as represented by a Markov chain of size 12. Tree-like tableaux provide a combinatorial formula for the PASEP.

Main Objective: Laborde Zubieta 2015

Conjecture: The number of corners in tree–like tableaux of size n is $n! \times \frac{n+4}{6}$.

Conjecture: The number of corners in symmetric tree–like tableaux of size $2n+1$ is $2^n \times n! \times \frac{4n+13}{12}$.

Permutation Tableaux

0	1	0	0	1	1
0	0	1	1		
0	1	1	1		
0					
1					

Figure: A permutation tableaux of size 12.

Definition

A permutation tableaux of size n is a Ferrers diagram of half-perimeter n such that

1. There is at least one 1 in every column.
2. There is no 0 with a 1 above it and a 1 to the left of it simultaneously.

The Bijection: (Aval, Boussicault, Nadeau 2011)

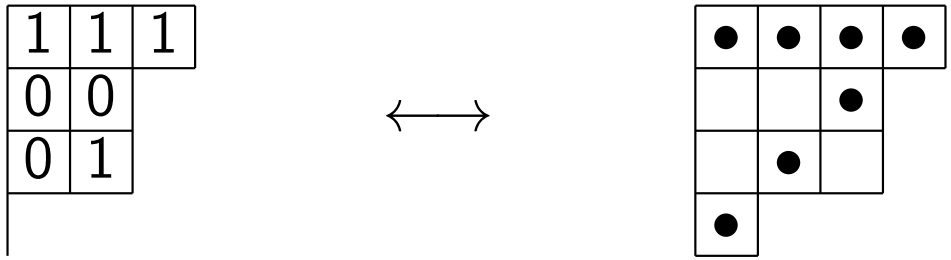


Figure: An example of the bijection

$$c(\mathcal{T}_n) = c(\mathcal{P}_n) + |\{P \in \mathcal{P}_n : \ M_n(P) = S\}| = c(\mathcal{P}_n) + (n-1)!$$

Theorem (Hitczenko, L.)

For permutation tableaux of size n,

$$\mathbb{E}_n C_n = \frac{n+4}{6} - \frac{1}{n}.$$

Symmetric Tree–Like Tableaux

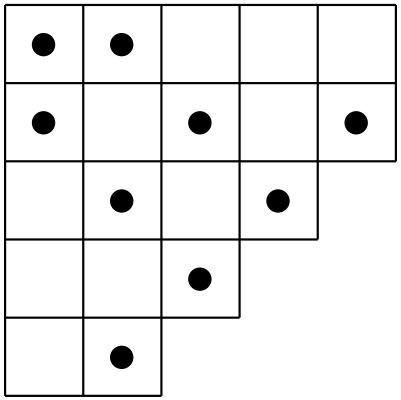


Figure: A symmetric tree–like tableaux of size 9. There are $2^n \cdot n!$ tableaux of size n.

Type-B Permutation Tableaux

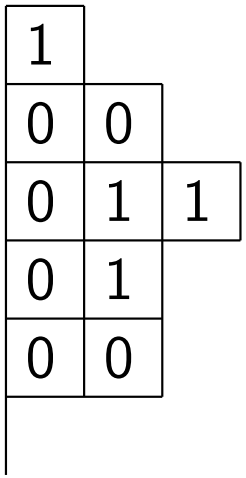


Figure: A type-B permutation tableaux of size 6.

Definition

A type-B permutation tableaux of size n is a shifted Ferrers diagram of half-perimeter n such that,

1. The rules of permutation tableaux are satisfied.
2. If there is a 0 on the diagonal, it is a 0-row.

The Bijection: (Aval, Boussicault, Nadeau 2011)

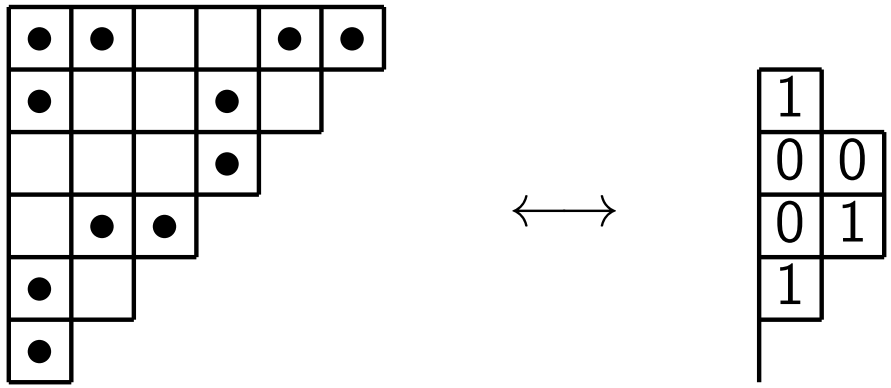


Figure: An example of the bijection between type–B permutation tableaux and symmetric tree–like tableaux.

1. Add a column and a root point then point unrestricted rows.
2. Replace all 0_R's with points (except on 0-rows).
3. Replace all non-diagonal 1_T's with points.
4. Make symmetric.

The Bijection

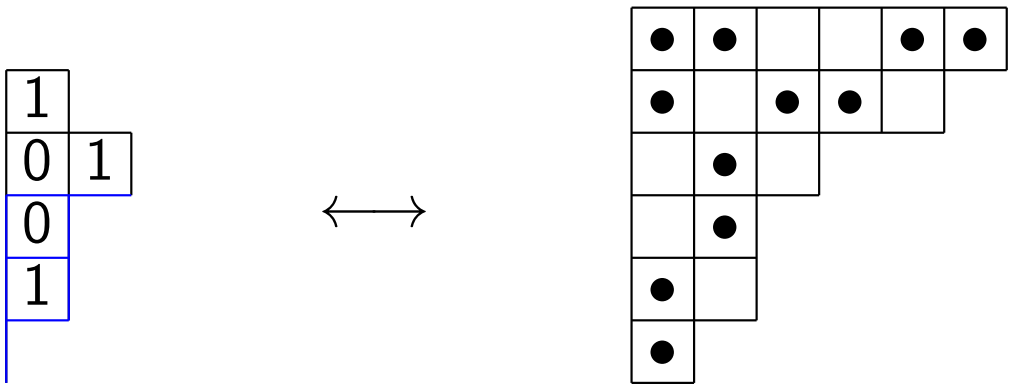


Figure: Transformation of the Shape

Key Relationship:

$$\begin{aligned} c(\mathcal{T}^{sym}_{2n+1}) &= 2c(\mathcal{B}_n) + 2|\{B \in \mathcal{B}_n : \ M_n(B) = S\}| + |\{B \in \mathcal{B}_n : \ M_1(B) = W\}| \\ &= c(\mathcal{B}_n) + 2^n(n-1)! + 2^{n-1}n!, \end{aligned}$$

Technique for proof: Corteel and Hitczenko 2007

If U_{n-1} is the number of unrestricted rows, then there are $2^{U_{n-1}+1}$ extensions.

$$\begin{aligned}
\mathbb{P}(U_n = U_{n-1}+1 | F_{n-1}) &= \frac{2}{2^{U_{n-1}+1}} = \frac{1}{2^{U_{n-1}}}. \\
\mathbb{P}(U_n = k | F_{n-1}) &= \frac{1}{2^{U_{n-1}+1}} \left(\binom{U_{n-1}}{k-1} + \binom{U_{n-1}}{k-1} \right) \\
&= \frac{1}{2^{U_{n-1}}} \binom{U_{n-1}}{k-1}.
\end{aligned}$$

Therefore,

$$\mathcal{L}(U_n | \mathcal{F}_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}, 1/2).$$

Relationship Between Measures

For all $B \in \mathcal{B}_{n-1}$,

$$\mathbb{P}_n(B) = \frac{2^{U_{n-1}(B)+1}}{|\mathcal{B}_n|} = 2^{U_{n-1}(B)+1} \frac{|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{P}_{n-1}(B)$$

Therefore, for any random variable X on $\mathcal{B}_{n-1}$,

$$\mathbb{E}_n(X) = \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})} X)$$

Simple Illustration

Proposition (known but new proof: Hitczenko, L.)

For all $n \geq 0$, $|\mathcal{B}_n| = 2^n n!$.

Proof can be deduced using the following:

$$\mathcal{L}(U_n|\mathcal{F}_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}, 1/2).$$

$$\mathbb{E}_n(X) = \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|}\mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})}X)$$

$$\mathbb{E}\left(a^{\mathrm{Bin}(m)}\right) = \left(\frac{a+1}{2}\right)$$

Corners in Type-B Permutation Tableaux

Theorem (Hitczenko, L.)

For type-B permutation tableaux of size n we have

$$\mathbb{E}_n C_n = \frac{4n+7}{24} - \frac{1}{2n}.$$

Proof.

$$\mathbb{E}_n \left(\sum_{k=1}^{n-1} I_{M_k=S, M_{k+1}=W} \right) = \sum_{k=1}^{n-1} \mathbb{E}_n \left(I_{M_k=S, M_{k+1}=W} \right)$$

Occupied Corners: Laborde Zubieta 2015

The generating polynomial $P_n(x) := \sum_{T \in \mathcal{T}_n} x^{oc(T)}$ satisfies the following recurrence:

$$\begin{cases} P_n'(x) = nP_{n-1}(x) + 2(1-x)P_{n-1}'(x) \\ P_0(x) = 1 \end{cases}$$

Theorem (Hitczenko, L.)

As $n \to \infty$, the limiting distribution of the number of occupied corners in a random tree–like tableau of size n is Pois(1).

Proof.

Theorem (Hitczenko, L.)

Let $P_n(x) = \sum_{k=0}^{m} a_{n,k} x^k$ *satisfy,*

$$P_n^{'}(x) = f_n(x) P_{n-1}(x) + g_n(x) P_{n-1}^{'}(x)$$

where $g_n(1) = 0$, $g_n'(x) = g_n = o(f_n)$, $f_n'(x) = 0$ *and*

$$f_n \frac{P_{n-1}(1)}{P_n(1)} \to c > 0, \quad as \quad n \to \infty.$$

If a sequence of random variables X_n *is defined by*

$$\mathbb{P}(X_n = k) = \frac{a_{n,k}}{P_n(1)} = \frac{a_{n,k}}{\sum_j a_{n,j}},$$

then

$$X_n \xrightarrow{d} \mathrm{Pois}(c) \quad as \quad n \to \infty.$$

Diagonal Boxes: Aval, Boussicault, Nadeau 2011

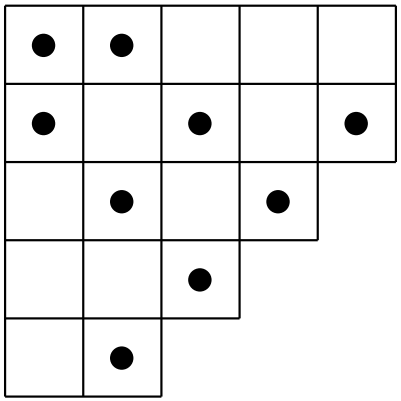


Figure: A symmetric tree–like tableaux with three diagonal boxes.

If $B(n,k)$ is the number of symmetric tree–like tableaux of size $2n+1$ with k diagonal cells, then $B_n(x) = \sum_{k=1}^{n+1} B(n,k)x^k$ satisfies the following recurrence,

$$\begin{cases} B_n(x) = nx(x+1)B_{n-1}(x) + x(1-x^2)B'_{n-1}(x), \\ B_0(x) = x. \end{cases}$$

Theorem (Hitczenko, L.)

Let D_n be the number of diagonal boxes in a random symmetric tree–like tableau of size $2n+1$. As $n \to \infty$,

$$\frac{D_n - 3(n+1)/4}{\sqrt{7(n+1)/48}} \xrightarrow{d} N(0,1).$$

Proof. Notice that

$$\mathbb{P}(D_n = k) = \frac{B(n,k)}{\sum_{k \geq 0} B(n,k)} = \frac{B(n,k)}{B_n(1)},$$

and then the conclusion will follow if the variance of $D_n \to \infty$ as $n \to \infty$ and $B_n(x)$ has non-positive real roots.

Proposition (Hitczenko, L.)

The variance of the number of diagonal cells in a random symmetric tree–like tableaux of size $2n+1$ is,

$$\operatorname{var}(D_n) = \frac{7(n+1)}{48}.$$

Proof. Straightforward computation of

$$\operatorname{var}(D_n) = \mathbb{E}(D_n)_2 - \mathbb{E}^2 D_n + \mathbb{E} D_n.$$

Proposition (Hitczenko, L.)

For all $n \geq 0$, the polynomial $B_n(x)$

a) *has degree $n+1$ with all coefficients non-negative, and*

b) *all roots real and in the interval $[-1, 0]$.*

Proof. Induction and Rolle's Theorem.
Recall:

$$\begin{cases} B_{n+1}(x) = (n+1)x(x+1)B_n(x) + x(1-x^2)B_n'(x), \\ B_0(x) = x. \end{cases}$$

Key Step:

$$B_{n+1}(x) = \frac{x(1-x^2)}{K_n(x)} \frac{d}{dx} [K_n(x)B_n(x)]$$

Future Directions

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3. Problems significant to PASEP