<u>**Fibre Bundles**</u>

A **<u>fibre bundle</u>** is a 6-tuple $(E, B, F, p, G, \{V_i, \phi_i\})$.

E is the **<u>total space</u>**, B is the **<u>base space</u>** and F is the **<u>fibre</u>**. $p : E \to B$ is the **<u>projection map</u>** and $p^{-1}(x) \approx F$. The last two elements of this tuple relate these first four objects.

The idea is that at each point of B a copy of the fibre F is glued, making up the total space E.

One way to do this is just take a product of B and F, i.e. let $E = B \times F$ and $p(x, y) = x$. For example $\mathbf{R^2} = \mathbf{R} \times \mathbf{R}$. This may be the first fibre bundle that anyone ever thought of ! (Descartes supposedly thought this way, according to S. S. Chern.)

It is not the case that every fibre bundle is of this form, i.e. **<u>trivial</u>**. To see this consider the Moebius strip and the cylinder as E where S^1 is the base B and an interval I is the fibre F.

Top Down

One way fibre bundles arise is top down, i.e. you start with E and break it up into fibres. For example if you have a Lie group G and a subgroup H you can form G/H. Then the fibre is H and the space is decomposed into copies of H, namely, the left cosets.

Example $R \to R/Z \approx S^1$

Example $R^2 \to R^2/Z^2 \approx S^2$, the torus.

Example $S^3 \to S^3/Z_n \equiv$ lens space.

Example $S^3 \to S^3/S^1 \approx S^2$.

These are special cases of what is known as a **group action**.

Bottom Up

We can also build bundles by taking a space and attaching things to it. The most natural example of this is the **tangent bundle** of a differentiable manifold. At each point of the manifold we glue all possible velocity vectors. This is the first example of a vector bundle, i.e. a fibre bundle where the fibre is a vector space (and the group G is the full linear group).

Another natural vector bundle occurs if we have a manifold embedded in a Euclidean space (or another manifold !). Then at each point we can attach all the vectors that are normal to that point. This forms the **normal bundle**.

Once you have one vector bundle you can make tons more. Suppose you have an operation that takes a vector space and gives you back a new one, like *. Then you can apply that operation to each fibre of you bundle to get a new one!

The Primordial Bundle-Soup

As I said above, Chern thinks that Descartes in some sense knew about fibre bundles. Who else did before the 20th century ? Probably Gauss had some idea (I have no hard evidence to support that). The reason I say this is because Gauss certainly thought a lot about tangent vectors, normal vectors and **ruled surfaces**, like $x^2 + y^2 - z^2 = 1$.

In group theory the idea of a ”twisted product” occurs pretty naturally and many people were certainly aware of it in the 19th century . The development of the theory of fiber bundles is tied to the the theory of groups partly because of this, but also because **homotopy theory** has to do with groups.

Sometime soon after the discovery of groups, it was observed that one could take the cartesian products of two groups to make a third. But, given a group G and a normal subgroup $H \subset G$ it wasn't always the case that $G \approx (G/H) \times H$. It fact this is pretty rare. So G was somehow broken up into copies of H, but those copies weren't glued together in an obvious way.

Now all of this is well understood because we have the notion of a **semi – direct product** of groups.

Fibre Bundles in the Pre-Cambrian

In 1934, Herbert Seifert published **The Topology of 3 – Dimensional Fibered Spaces**, which contained a definition of an object that is a kind of fibre bundle. Seifert was only considering circles as fibres and 3-manifolds for the total space. The point was that 2-manifolds had been classified and now everyone was trying to classify 3-manifolds. The idea was to decompose a 3-manifold into circles over an ”orbit surface”. This seemed like a reasonable approach since surfaces were classified.

His definition of a **fibered space** is that it is a 3-manifold satisfying the following:

1) The manifold can be decomposed into fibers, where each fiber is a simple closed curve.

2) Each point lies in exactly one fibre.

3) For each fiber H there exists a *fiber neighborhood*, that is, a subset consisting of fibres and containing H, which can be mapped under a fiber-preserving map onto a fibered solid torus, where H is mapped onto the ”middle fibre”.

Other things related to Fibre Bundles

After the invention of **Lie groups** people were taking quotients like crazy (and still are). They found a lot of funny things happening.

For example, Hopf found an interesting map $S^3 \to S^2$ which was homotopically non-trivial. This map, known as the Hopf fibration, has many interesting properties and still plays a major role in differential geometry and algebraic topology. It is in fact a fibre bundle with base S^2, total space S^3 and fibre S^1 ! In fact the map could be viewed as a quotient map $S^3 \to S^3/S^1$.

The reason that the Hopf fibration is called the Hopf fibration and not the Hopf fibre bundle is that there is something called a **fibration**, which is a lot like a fibre bundle but more general. A fibration consists of two spaces E and B and a map $p : E \to B$ with a property called the homotopy lifting property. There is a rough notion of a fibre that can be made for fibrations, but they are a hard to do calculus on.

There is also object known as a **sheaf** which also follows this idea of gluing a space to each point of another space. These were invented in the 1950's and developed extensively in the 1960's, solving many problems in algebraic geometry. The example of a sheaf of germs of functions on a differentiable manifold is a familiar example of a sheaf.

Some Applications

There are too many mathematical applications to list here!.

In physics the idea of a tangent bundle is very useful. In mechanics, for example, often the configuration space of a system is a manifold, and the phase space is the tangent bundle.

There have been other uses of fibre bundles in physics. They are used for lots of things physics, such as gauge theory, which was invented by Hermann Weyl.

In the late 1970's fibre bundles became quite fashionable in condensed matter physics for explaining certain properties of defects in ordered matter, like a crystal. It was apparently quite successful in explaining some of the behavior of **Superfluid helium – 3**. Some people that I know study liquid crystals and sometimes use fibre bundles. I have heard it said that the strength of certain steels was increased due to defects, and that this could be explained with bundles.

Vibrational modes of molecules (buckybulls work well due to their symmetries) have are being analyzed using vector bundles. There are some intriguing relations here between continuous and discrete mathematics (graph theory).

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Group Actions

If X is a set and G is a group then a **group action** (on the left) is a map $\cdot : G \times X \to X$ that satisfies

(i) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1 g_2 \in G$, $x \in X$,
and
(ii) $e \cdot x = x$ for all $x \in X$, where e is the identity of G.

Additionally we define the **orbit** of a point x of X as

$orb(x) = \{gx \mid g \in G\}$

and the **stabilizer** of x as

$Stab(x) = \{g \in G \mid gx = x\}$.

These ideas are EXTREMELY useful.

Remark
The orbits are equivalence classes and one should try to picture the corresponding quotient space.

Transitive Actions

Example The symmetry group of a cube acts on the cube. The symmetry group is isomorphic to A_4, a group of 24 elements. What are the orbits and stabilizers ?

Remarks
1) Orbits are disjoint.
2) X is the union of the orbits.

Example Let $G = O(n)$ and $X = S^n$.

Definition The action of G on X is said to be **transitive** if for all $x, y \in X$ there is a $g \in G$ such that $gx = y$. In other words, there is only one orbit.

Example $R \times R^2 \to R^2$, $(x,(a,b)) \mapsto (a+x,b)$.

Remark
If "dim G" $<$"dim X" then one expects the action not to be transitive.

Faithful Actions

Definition The action of G on X is said to be **faithful** if $gx = hx$ for all $x \in X$ implies that $g = h$.

Or: the action defines a map $G \to X^X$. Then the action is faithful if this map is 1-1.

Example $\mathbf{C}^* \times \mathbf{S^1} \to \mathbf{S^1}$, where $(re^{i\theta}, z) \to e^{i\theta} z$.

Remark
If $G \subset X^X$ and G acts by $(f, x) \to f(x)$ then the action is faithful.

Free Actions

Definition The action of G on X is said to be **free** if for all $g \in G$, $g \neq e$, and for all $x \in X$, $gx \neq x$.

Example $R \times R^2 \to R^2$ as above is a free action.

Example The group of deck transformations of a covering acts freely on the top space of the covering.

Example Let $G = O(n)$ and $X = S^n$. Then the obvious action is not free.

Example The action of the symmetry group of a cube on the cube. This action is not free. But it is **fixed – point free**, i.e. there is no point that is fixed by all the elements of the group.

<u>The Fundamental Theorem</u>

<u>Theorem</u> Suppose that G acts on a set X and $x \in X$. Then there is a bijection $\Phi : G/Stab(x) \to orb(x)$.

<u>Remarks</u>
1) It is often the case that Φ will have many good properties.
2) If the action is transitive then $orb(x) = X$. This is of particular interest.

Examples of Transitive actions

Example $O(n+1) \times S^n \to S^n$.

Example $SO(n+1) \times S^n \to S^n$

Example $U(n) \times S^{2n-1} \to S^{2n-1}$

Example $SU(n) \times S^{2n-1} \to S^{2n-1}$

Example Take $G = O(n)$ acting on the set of lines through the origin in R^n, i.e. RP^n.

Example Take $G = U(n)$ acting on the set of lines through the origin in C^n, i.e. CP^n

Definition Let $G_{n,k}(V)$ be the set of k-dimensional subspaces of a vector space V.

Example Take $G = O(n)$ acting on $G_{n,k}(R^n)$. The stabilizer of a point is isomorphic to $O(n-k)$.

Non-transitive actions

Example For any group G and subgroup $H \subset G$, let H act by usual left multiplication. In particular we could take G to be the additive group of the reals and H to be the integers $\mathbf{Z}$.

Example Any group G acts on itself by conjugation. If a subgroup is invariant under this action, then it is normal. In that case the action of G on itself induces an action of G on the normal subgroup.

Example S^1 acts on S^2 by rotation. What is the quotient space ?

<u>Lie Groups and Bundles</u>

<u>Theorem</u> Let G be a Lie group and let H be a closed subgroup. The projection $\pi : G \to G/H$ gives rise to a fibre bundle where H is the fibre *and* the structure group.

<u>Remarks</u>
1) This is an example of a **<u>principle bundle</u>**.

2) This is a special case of a group acting transitively on a space and thus it gives rise to a fibre bundle. Here G acts on G/H transitively and the isotropy group (stabilizer) is H.

A sample bundle

Example $SU(n) \times CP^{n-1} \to CP^{n-1}$. This action is transitive. What is the isotropy group ? If a matrix in $SU(n)$ fixes a line then it has the form

$$\begin{pmatrix} & & & & 0 \\ & & & & . \\ & & A & & . \\ & & & & . \\ 0 & . & . & . & \alpha \end{pmatrix}$$

where A is in $U(n-1)$ and $det(A) = \alpha$. Therefore we have the bundle

$$U(n-1) \to SU(n) \to CP^{n-1}.$$

The Hopf Fibration

In the special case n=2 of the the above bundle we have

$$U(1) \to SU(2) \to CP^1.$$

But each of the three spaces is homeomorphic to a sphere : $U(1) \approx S^1, SU(2) \approx S^3$ and $CP^1 \approx S^2.$ So what does this bundle have to do with

$$S^1 \to S^3 \to S^3/S^1$$

(where we view S^3 as a group via the quaternions), ? It turns out that they are the same bundle.

Another View

There is yet another way to look at the Hopf fibration:

$$U(1) \to SU(2) \to SU(2)/U(1).$$

Lets look at this carefully. U(1) is a group of 1×1 matrices each of whose determinant has absolute value 1, so that is clearly S^1.

$$SU(2) = \{ \left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right) \mid \mid z \mid^2 + \mid w \mid^2 = 1 \}$$

,

and if we write $z = x + iy$ and $w = a + bi$ it is easy to see that $x^2 + y^2 + a^2 + b^2 = 1$, so $SU(2) \approx S^3$. We will view $U(1)$ as a subgroup of $SU(2)$ by the imbedding $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$.

Getting to S^2

Let $T = \{A \in SU(2) \mid tr(A) = 0\}$. The typical element of T has the form $\begin{pmatrix} xi & w \\ -\bar{w} & -xi \end{pmatrix}$, where x is real and $w = a + bi$ is complex. Since this matrix is special unitary, $x^2 + a^2 + b^2 = 1$, i.e. $T \approx S^2$. Define $\pi : SU(2) \to T$ by the equation

$$\pi(P) = PEP^*,$$

where $E = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The fibers of π are exactly the cosets of $SU(2)/U(1)$! So we have

$$\begin{array}{ccccccc} U(1) & \to & SU(2) & \longrightarrow & T & \approx & S^2 \\ & & & \searrow & \updownarrow & & \\ & & & & SU(2)/U(1) & & \end{array}.$$

There is also a direct way to write down a map from S^3 to S^2.

Back to Bundles

A bundle with fibre F, total space E and base space M is a map

$$\pi : E \to M$$

where each point p has an open neighborhood $U \subset M$ such that $\pi : \pi^{-1}(U) \to U$ is the projection pr_1 up to diffeomorphism. We say the bundle is locally trivial and that Φ is a local trivialization. Also, in general it is assumed that all of the spaces involved are C^∞ manifolds and that all the maps are smooth.

Changing Coordinates

Observe that Φ is like a coordinate chart on a manifold. Suppose that Φ_1 and Φ_2 are local trivializations for U_a and U_b where $U_a \cap U_b \neq \phi$. Then each $p \in U_a \cap U_b$ determines a diffeomorphism from F to F:

$$\varphi_{ba} : U_a \cap U_b \to Diff(F)$$

where φ_{ba} is defined by

$$\Phi_b \circ \Phi_a^{-1}(p, f) = (p, \varphi_{ba}(p)(f)),$$

with $(p, f) \in (U_a \cap U_b) \times F$.

Remarks

1) $\varphi_{aa} = id_F$.
2) If $p \in U_a \cap U_b \cap U_c$ then $\varphi_{cb}(p) \circ \varphi_{ba}(p) = \varphi_{ca}(p)$.
3) The φ_{ba} are known as **transition functions**.

Building a Bundle

One may reconstruct $\pi : E \to M$ given the open sets and the transitions functions and the fibre. First consider the disjoint union $\bigsqcup_{\alpha} U_{\alpha} \times F$. Then glue together (p_a, f_a) and (p_b, f_b) if $p_a = p_b = p$ and $f_b = \varphi_{ba}(p)(f_a)$.

Two Bundles

If the transition functions all lie in a group $G \subset Diff(F)$ then G is said to be the **structure group** of the bundle.

Remark A bundle can have many structure groups.

Definition $\pi : E \to M$ is a **vector bundle** if F is a vector space V and the structure group $G \subset Gl(V)$, i.e. G is a subgroup of the group of invertible linear maps.

Definition $\pi : P \to M$ is a **principle G-bundle** if the fibre F is a Lie group G and the action of the stucture group on G coincides with the action of a subgroup of G on G by left multiplication.

Example $S^1 \to S^1, z \mapsto z^2$ is a principle bundle with fibre Z_2.

New Bundles from old

Example If $E \to M$ is a vector bundle then we can form the bundle of bases $B(E) \to M$. How do we do this?

In general, given a bundle $E \to M$ one may construct a new bundle, where the fibre of the original bundle, E, is just replaced by the structure group G. We do this by taking the disjoint union $\coprod_\alpha U_\alpha \times G$ and glueing together (p_a, h_a) and (p_b, h_b) if $p_a = p_b = p$ and $h_b = \varphi_{ba} \cdot h_a$.

Principle Bundles[1]

A **G-principle bundle**, or simply a **principle bundle** is a fibre bundle $\pi : P \longrightarrow E$ with fibre F equal to the structure group G and having the property that for all U_a and U_b with $U_a \cap U_b \neq \phi$,

$$\varphi_{ba} : U_a \cap U_b \to Left(F) \subset Diff(F),$$

where $Left(F) = \{L_g \mid L_g(h) = gh, \forall h \in G, g \in G\}$. In other words, changing coordinates corresponds to multiplying the fibre on the left by some element of G.

Lemma For every G-principle bundle, G acts naturally on P on the right.

Proof Given $u \in P$, we want to define ug, for each $g \in G$. Let U be a neighborhood about $\pi(u)$ that has a trivialization. Using these coordinates, represent u as $(\pi(u), h)$ where $h \in G$. Then define ug to be the point of P that has the coordinates $(\pi(u), hg)$. It is not hard to check that this definition is independent of coordinates, and then it is clear that it is a right action.

[1] These notes are taken essentially from "Metric Differential Geometry" by Karsten Grove.

Example of Principle Bundles

Example The projection $S^n \longrightarrow RP^n$. Here $G = O(1) = Z_2$.

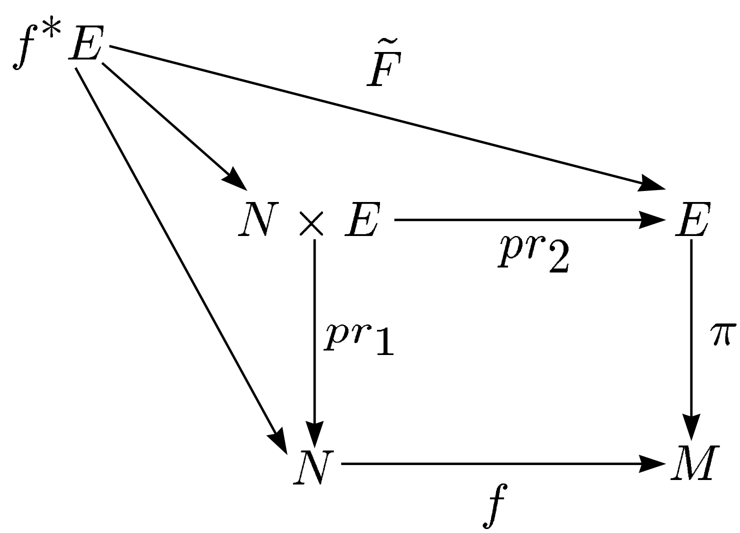
Example The Hopf map $S^{2n+1} \longrightarrow CP^n$. Recall that CP^n is the set of all lines in C^n. A point on S^{2n+1} gets sent to the complex line that contains it. Here $G = U(1) = S^1$.

Example $\tilde{X} \longrightarrow X$ where $\tilde{X}$ is the universal covering space of X. Then $G = \pi_1(X)$.

Example The covering need not be universal. Take $S^1 \longrightarrow S^1$ by $z \mapsto z^2$. This is a principle bundle with fibre Z_2. What is the general situation ?

Pullbacks of Bundles

Let $E \longrightarrow M$ be a bundle with fibre F and structure group G. Consider a map $f : N \longrightarrow M$, where N is any smooth manifold. Define the pullback of $E \longrightarrow M$ by f to be the bundle $f^*E \longrightarrow N$ with fibre F where $f^*E = \{(q,u) \mid f(q) = \pi(u)\}$ and the projection map is defined by $(q,u) \mapsto \pi(u)$. In terms of diagrams we have



Parallel transport along a curve $\gamma : [a,b] \longrightarrow M$ may be described using the pullback γ^*E.

Connections in Bundles

Let $\pi : E \longrightarrow M$ be a bundle. The subbundle (a distribution) $V \longrightarrow E$ of the tangent bundle $TE \longrightarrow P$ defined by

$$V = \{X \in TE \mid \pi_*(X) = 0\}$$

is called the **vertical bundle** of E.

At this point we introduce the notation E_p to mean $\pi^{-1}(p)$, $p \in M$.

With this in mind, $V_u \subset T_u E$ and V_u is actually the tangent space at u of the fibre $E_{\pi(u)}$. But there is no canonical complement to H_u to V_u, i.e. a subspace $H_u \subset T_u E$ such that

$$V_u \oplus H_u = T_u E.$$

Such a space is called a **horizontal space** at u.

Definition A **connection** in E is a subbundle of the tangent bundle of E such that each fibre is a horizontal space. If E is a principle bundle with group G we require that the connection be G-invariant, i.e.

$$H_{ug} = (R_g)_* H_u, u \in E, g \in G.$$

Connections in Principle Bundles

If we have a connection in a principle bundle $P \longrightarrow M$ then a choice of a horizontal subspace at u is equivalent to a choice of a projection

$$proj_u : T_uP \longrightarrow V_u.$$

Then vertical space (i.e. a fibre of the vertical bundle) my be viewed as the tangent space to the fibre, G. In turn, this can be viewed as g, the lie algebra of G, so we really have

$$proj_u : T_uP \longrightarrow g.$$

Therefore a connection is equivalent to having a g valued 1-form on P which is invariant under the right action of G.

Connections in Vector Bundles

Let $E \longrightarrow M$ be a vector bundle. We define a differential operator ∇ as follows. Identify V_u with the fibre $E_{\pi(u)}$. Let $\eta \in \Gamma(E)$ be a section, i.e. $\eta : M \longrightarrow E$ and $\pi \circ \eta = id_M$. Then let

$$\nabla\eta : T_pM \longrightarrow E_p$$

be defined for each $p \in M$ by

$$\nabla\eta(X) = \nabla_X\eta = \eta_*(X)^v = \eta_*(X) - \eta_*(X)^h.$$