1. GLOBAL SIGNATURES FOR ROBOT CONTROL AND RECONSTRUCTION

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Abstract

We address the problem of control-based recovery of robot pose and the environmental lay-out. Panoramic sensors provide us with a 1D projection of characteristic features of a 2D operation map. Trajectories of these projections contain the information about the position of a priori unknown landmarks in the environment. We introduce here the notion of spatiotemporal signatures of projection trajectories. These signatures are global measures, characterized by considerably higher robustness with respect to noise and outliers than the commonly applied point correspondence. By modeling the 2D motion plane as the complex plane we show that by means of complex analysis the reconstruction problem is reduced to a system of two quadratic equations in two variables.

1.1 Introduction

Consider the following situation. We are given three landmarks in the plane, which we represent as the complex numbers \( z_1, z_2 \) and \( z_3 \), and a sensor that can measure the angle between any pair of these points, from any position in the plane at which it is placed. Let \( \theta \) denote the angle between \( z_1 \) and \( z_2 \) and let \( \phi \) denote the angle between \( z_2 \) and \( z_3 \). If the sensor undergoes a circular motion (the left of figure 1.1) we may record the angles and then plot \( \phi \) versus \( \theta \). For example, in in the right of figure 1.1 we see such a graph for the points \( z_1 = 2i, z_2 = 2 + 2i \) and \( z_3 = 5 \).

The primary motivation for this paper is to investigate the extent to which the curve displayed in the right of figure 1.1 characterizes the three points. In other words, we are want to solve the inverse problem: given the data depicted in the right of figure 1.1, how can the scene be reconstructed? (Generally, a Cartesian frame is not available, so what we seek are the magnitudes of \( z_1, z_2 \) and \( z_3 \), and the angles between them with respect to the origin, which we take to be the center of the circular motion.) Note that while we require to the sensor motion to be constrained to a circle, there is no reference made to where on the circle the sensor
Figure 1.1: The problem: find \( z_1, z_2 \) and \( z_3 \) by moving on a circle and measuring the angles \( \theta \) and \( \phi \) during the motion. On the left we see the actual motion of the sensor, while on the right we see a plot of \( \phi \) vs. \( \theta \) for \( z_1 = 2i, z_2 = 2 + 2i, z_3 = 5 \).

is at any time. Rather, we take as our hypothesis that we are densely sampling the angles, although not necessarily at a uniform rate.

### 1.1.1 Problem Statement

We begin with the general form of our problem. Suppose that \( z_1, \ldots, z_n \) are complex numbers representing \( n \) landmarks in the plane, and a sensor is available with the ability to measure the angles between the landmarks with respect to any given position of the sensor, \( w \) (i.e. it can measure the angle \( \angle z_k w z_l \), \( k, l = 1, \ldots, n \)). Then for \( m \) positions \( w_1, \ldots, w_m \) of the sensor\(^1\), can one determine \( z_1, \ldots, z_n \) and \( w_1, \ldots, w_m \) given only the measurements \( \angle z_k w_j z_k, k = 1, \ldots, n, j = 1, \ldots, m \)?

One approach to the above is to write down all of the trigonometric equations associated with the configuration. Suppose that \( \chi_{k,j} \) is the angle between the segment connecting \( z_k = x_k + iy_k \) and the horizontal line through \( w_j = a_j + ib_j \) (see figure (1.1.1)). What may be measured by the sensor, i.e. the known values in the problem, are the differences \( m_{k,j} = \chi_{k,j} - \chi_{1,j} \). From figure (1.1.1) we see that we have the equation

\[
\tan(\chi_{k,j}) = \frac{y_k - b_j}{x_k - a_j},
\]

so that

\[
m_{k,j} = \arctan \left[ \frac{y_k - b_j}{x_k - a_j} \right] - \chi_{1,j},
\]

where \( k = 2, \ldots, n, \ j = 1, \ldots, m \). The above non-linear system consists of \( 2m + 3m \) unknowns and \( m(n - 1) \) equations. If we in addition require that the \( w_j \) all lie on a circle then we have the additional equations \( a_j^2 + b_j^2 = 1, \ j = 1, \ldots, m \), making for a total of \( mn \) equations. In general the system is over-constrained, which may be

\(^1\)In our model of the panoramic sensor, the pose of the sensor means only its position, and not orientation. For real applications it is possible to define and compute and orientation with our method if it is needed.
Figure 1.2: Here we have the sensor placed at the point \( w_j = a_j + ib_j \) in the presence of landmarks at \( z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n \). The angle formed between the segment connecting the landmark at \( z_k \) and the sensor at \( w_j \), and the horizontal line through \( w_j \) is denoted at \( \chi_{k,j} \). This is not an angle that our sensor can measure. Rather, the sensor provides the angles between the landmarks, since it can does not have a means of detecting an absolute direction, i.e. it does not have a compass. Thus it measure the angles \( \chi_{k,j} - \chi_1, j \).

helpful because in a experiment the angular measurements will of course contain errors. One may try to find an solution in the “least squares sense” by minimizing the sum of the squares of these equations. This method has numerous problems. Most prominent is that one may not find the global minimum of the function being minimized. Additionally, this approach is computationally expensive.

In this paper we are primarily concerned with a special case of the above problem. We will assume that the first two points, \( z_1 \) and \( z_2 \), are known and that the points \( w_i \) all lie on a circle, i.e. \( |w_i| = 1, i = 1, \ldots, m \). Additionally, we will assume that the \( w_i \) are densely distributed in the circle (but not necessarily in a uniform fashion). Notice that if the problem can be solved for \( n = 3 \) with \( z_1, z_2 \) known, then it can be solved for any \( n > 3 \), as long as \( z_1, z_2 \) are known. To do this, use the method used to find \( z_3 \) given \( z_1 \) and \( z_2 \), but instead using any \( z_j \) in place of \( z_3 \). From an experimental point of view, this means we may reconstruct an entire scene if we lay down two known reference points.

### 1.1.2 Signatures

In the general problem stated above, we are given measurements \( m_{2,1}, \ldots, m_{n,m} \) and we want to solve a system of equations

\[
F_1(m_{2,1}, \ldots, m_{n,m}, z_1, \ldots, z_n, w_1, \ldots, w_n; \chi_{1,1}, \ldots, \chi_{1,m}) = 0
\]

\[
\vdots
\]

\[
F_m(m_{2,1}, \ldots, m_{n,m}, z_1, \ldots, z_n, w_1, \ldots, w_n; \chi_{1,1}, \ldots, \chi_{1,m}) = 0
\]

where the unknowns are \( z_1, z_2, \ldots, z_n \), and \( w_1, w_2, \ldots, w_n \).
The method we describe below allows for the equations to be decoupled, (unlike the above system), resulting in a single equation of the form

$$G(f(m), z_1, z_2, z_3) = 0$$

where $m = (m_{2,1}, \ldots, m_{2,m}, m_{3,1}, \ldots, m_{3,m})$ and $f$ is a smooth complex valued function defined on the space that $c$ lives in. We will take $f$ to be the composition of an integral operator with an analytic function that acts pointwise on $m$. We say that $f(m)$ is a signature of $(z_1, z_2, z_3)$. The fact that $f$ is smooth makes the system robust to errors in measurement, while taking advantage of global information that may be embedded in the measured data.

### 1.1.3 Panoramic Sensors

Recently, many researchers in the robotics and vision community have begun to investigate the use of curved mirrors to obtain panoramic and omni-directional views. Typically, such systems consist of a standard CCD camera pointing upward at a convex mirror, as in figure 1.3. How to interpret and make use of the information obtained by such sensors, e.g., how it can be used to control robots, is not immediately clear. It does seem likely that panoramic systems may be able to handle some problems that are difficult or impossible to address with a standard camera.

A panoramic sensor provides a natural means of addressing the above problem experimentally, since it acts very much like an overhead camera. A natural feature to consider for extraction from an image taken on such a system is any edge that is vertical in the real world, because these edges appear as radial lines in the image (this is clear in figure 1.3). In the center of this image the lens of the camera is clearly visible. To measure the angles between the vertical edges we extend them to the center of the lens and compute the angle at their intersection.
1. **GLOBAL SIGNATURES**

1.1.4 **Overview of the Solution**

For each point on the circle, \( z \), we have two real numbers \( \theta(z) \) and \( \phi(z) \). (These are the values that are measured experimentally.) Thus we have a function from the unit circle to the plane. The area bounded by this curve depends of course on \( z_1, z_2 \) and \( z_3 \), i.e. we can consider it as a function \( A(z_1, z_2, z_3) \), and in this case this number is the signature discussed above.

Using a panoramic sensor as described above, the curve can be sampled experimentally. Hence the signature, \( m \), can be computed numerically, which is equal to \( A(z_1, z_2, z_3) \) can be calculated. On the other hand, a general formula for \( A(z_1, z_2, z_3) \) can be determined using the methods of complex analysis, and so the equation \( A(z_1, z_2, z_3) = m \) gives a relationship between \( z_1, z_2 \) and \( z_3 \). If \( z_1 \) and \( z_2 \) are known then the equation \( A(z_1, z_2, z_3) = m \) can be solved for \( z_3 \). Because the angles are integrated to calculate \( m \), the value of \( z_3 \) obtained from solving the equation is very insensitive to errors in the original measurements.

We can make a slight modification to the above procedure that simplifies the equations greatly. The integral that represents the average of the angles is not a simple one. A much better quantity (for reasons discussed in section 1.3.2) to consider is the average of certain complex exponentials of the angles. This poses no experimental difficulty - one simply takes the measured angles and applies the appropriate transformation to them before averaging.

1.1.5 **Related Work**

There are several works from the eighties - early nineties which perform reconstruction using vertical edges and known motion (Kriegman [7], Kak [6]). There is work in structure from motion from circular trajectories (Shariat and Price [12], Sawhney [11]). This work, however, uses a set of equations with the constraint that the projection centers are on a circle.

Recently, a number of approaches to navigation and reconstruction using omnidirectional systems have been proposed (Nayar [8], Svoboda [14], Onoe [9], Srinivasan [2], Yagi [15], Medioni [13]). The work by Yagi and Medioni is very similar but uses an already known environmental map. The work most relevant to this paper is that done on omnidirectional multibaseline stereo done by Kang and Szeliski [5], but this work uses conventional cameras and measurement equations.

1.1.6 **Contributions**

Our reconstruction method has a number of unusual features that the standard methods do not.

1. Rather than constructing one or more equations for each measurement, we first process the measured data to produce a single number, which is then used to create exactly two equations in the two real variables representing the coordinates of the unknown landmark. **These equations are quadratic** (see equation 1.8).

2. Due to the simplicity of the equations, optimization methods are not needed, and when the solutions are found they are known to be true solutions, as opposed to possible spurious solutions produced by numerical optimization methods.
(3) The positions of the camera never appear as variables, although they can be reconstructed after the position of the unknown landmark is determined.

(4) Due to the stable means of processing the data (integration), our method handles noise well. In addition, if one desired to carry out a detailed error analysis, it is possible since the equations are exactly solvable (this is the topic of ongoing work). This is not the case with a method that produces a very large number of nonlinear equations.

1.2 Applications to Robotics

The motivation for the above is the problem of having a robot enter an unknown environment and use unknown landmarks to estimate its pose.

Consider the a 2D situation where a robot can detect and track three fixed points in the plane and measure the angles $\theta$ and $\phi$, between them, as in figure 1.4.

Figure 1.4: The angles between three fixed landmarks provide a natural coordinate system

In the formal sense, $\theta$ and $\phi$ define a local coordinate system on the plane, and so even if the robot does not have a means of measuring its Cartesian coordinates with respect to some frame, it can measure it’s $(\theta, \phi)$ coordinates.

To estimate the position of the robot one must convert from angular coordinates to Cartesian coordinates. (Of course, to do this one must know the positions of the landmarks.) This is an old problem in fact, familiar to sailors and surveyors. A method used by surveyors can be found in [3]. One approach is simply to write down the equation of each of the three circles that contain the robot and two of the landmarks and the position of the robot is where they all intersect. Despite the fact that this method does not provide an explicit formula for converting from angular to Cartesian coordinates, it is effective from a numerical viewpoint. This problem and the effects of noise on it is considered in [1]. A fast linear algorithm is
given in [4]. In [10] this method of robot pose estimation is studied experimentally using a panoramic sensor constructed from a conical mirror.

Thus, if a robot is able to calculate the position of three landmarks, it can use this information for future pose estimation. In the next section we demonstrate our method for finding the positions of unknown landmarks given that two known reference points are available.

1.3 Calculating Signatures

1.3.1 Angles and Complex Logarithms

If \( z \) is a complex number, then there are real numbers \( \Theta \) and \( r \) such that \( z = re^{i\Theta} \). Even if \( r > 0 \), the number \( \Theta \) is not uniquely determined - adding any multiple of \( 2\pi \) to \( \Theta \) will give the same value of \( z \). For a given \( z \) the set of all such angles is called the argument of \( z \) and denoted \( \arg(z) \). Note that \( \arg \) is not a real valued function in the usual sense, but up to an additive factor of \( 2\pi \), “the” argument of a complex number is well defined\(^2\). If \( z > 0 \) we define the logarithm of \( z \) by

\[
\log(z) = \ln(r) + i \arg(z),
\]

where \( \ln \) is the real natural logarithm. Therefore, \( \Theta = \Im(\log(z)) \) and since \( \Im(w) = \frac{\Im(z)-\Im(w)}{2i} \) for any \( w \), we have that \( \Theta = \frac{\log(z) - \log(w)}{2i} = \log(\frac{z}{w}) \). This allows us to define the angle between the complex numbers \( z_1 \) and \( z_2 \) by

\[
\angle(z_1, z_2) = \frac{1}{2i} (\log(\frac{z_1}{z}) - \log(\frac{z_2}{z})) = \frac{1}{2i} \log(\frac{z_1 z_2}{z_2 z_1}).
\]

1.3.2 Applying the Residue Theorem

We denote the unit circle in the complex plane as \( S^1 = \{e^{it} | t \in [0, 2\pi] \} \). Let \( z_1, z_2 \) and \( z_3 \) be complex numbers. Then we define the angle between \( z_1 \) and \( z_2 \) with respect to \( z = z(t) = e^{it} \) as

\[
\theta(z) = \frac{1}{2i} \log \left[ \frac{(z_1 - z(t))(z(t) - z_2)}{(z(t) - z)(z_2 - z(t))} \right]
\]

and likewise we define the angle between \( z_2 \) and \( z_3 \) with respect to \( z = z(t) = e^{it} \) as

\[
\phi(z) = \frac{1}{2i} \log \left[ \frac{(z_2 - z(t))(z(t) - z_3)}{(z(t) - z)(z_3 - z(t))} \right].
\]

Notice that \( z = \frac{1}{z} \) because \( z = e^{it} \), so that in fact

\[
\theta(z) = \frac{1}{2i} \log \left[ \frac{(z_1 - z)(z(t) - \frac{1}{z_2})}{(z(t) - \frac{1}{z})(z_2 - z)} \right]
\]

and

\(^2\)This makes for certain technical problems when computing with the logarithm, but we will for simplicity ignore them since they will have cause no inconsistencies in our calculations.
\[
\phi(z) = \frac{1}{2i} \log \left[ \frac{(z_2 - z)(\overline{z_3} - \frac{1}{z})}{(z_2 - \frac{1}{z})(z_3 - z)} \right].
\]

(1.5)

Considered as functions of \( z \), equations (1.4) and (1.5) define functions which are analytic at all points of their domain. This is a necessary hypothesis in our next step, in which we apply the residue theorem.

A natural quantity to compute is

\[
\oint_{S^1} \theta d\phi,
\]

which represents the area of the curve discussed in section 1.1.4. Unfortunately, calculating this integral is problematic due to the so-called “branch cut” of the complex logarithm. The origin of this difficulty is due to the ambiguity in the choice of angle in the polar representation. But the ambiguity disappears of course if one applies to the logarithm an exponential function. Therefore a reasonable form to integrate would appear to be \( e^{2\omega} d\phi \). From an experimental point of view, what this means is that the angular data is gathered in the form of two lists of angles, and then one of these lists is transformed by the above function and then the two lists are used to calculate the integral. Since \( d\phi = \frac{d\omega}{dz}dz \) the integral can be written as the complex contour integral

\[
\oint_{S^1} e^{2\omega} \frac{d\phi}{dz} dz.
\]

(1.7)

Clearly

\[ e^{2\omega} = \frac{(z_1 - z)(\overline{z_2} - \frac{1}{z})}{(z_1 - \frac{1}{z})(z_2 - z)} \]

and

\[ \frac{d\phi}{dz} = \frac{1}{2i} \left[ \frac{(\overline{z_2} - \frac{1}{z})(z_3 - z)}{(z_2 - \frac{1}{z})(\overline{z_3} - \frac{1}{z})} \right] \cdot \frac{d}{dz} \left[ \frac{(z_2 - z)(\overline{z_3} - \frac{1}{z})}{(z_2 - \frac{1}{z})(z_3 - z)} \right] \]

and so the integrand in the above integral is a rational function in \( z \). Although in principle it is possible to compute the residues of this by hand, for this calculation and another below we found it useful to employ the symbolic computing package Maple to find the location of the singularities of this function and compute their residues. Singularities were found to occur at \( \frac{1}{z_1}, \frac{1}{z_2}, z_2 \) and \( z_3 \).

For completeness we now state the residue theorem.

**The residue Theorem** Let \( C \) be a positively oriented simple closed contour within and on which a function \( f \) is analytic except at a finite number of singular points \( w_1, w_2, \ldots, w_n \) interior to \( C \). If \( R_1, \ldots, R_n \) denote the residues of \( f \) at those respective points then

\[
\int_C f(z) dz = 2\pi i (R_1 + \cdots + R_n).
\]

Thus the value of (1.7) depends on whether or not \( z_1, z_2 \) and \( z_3 \) lie inside or outside of \( S^1 \). Assuming that the three landmands lie outside of the circle implies that the contour encloses the singularities at \( \frac{1}{z_1} \) and \( \frac{1}{z_3} \). If \( z_1, z_2 \) or \( z_3 \) lie on \( S^1 \)
1. **GLOBAL SIGNATURES**

then this method does not apply.) By using Maple we have that the value of (1.7) is

\[
\pi (\bar{z}_1 z_2 \bar{z}_2 z_2 z_3 + 2 \bar{z}_3 z_2 z_2 - \bar{z}_1 \bar{z}_2 z_3 z_3 - z_3 \bar{z}_3 \bar{z}_2 z_2 + z_3 \bar{z}_2 + \\
\bar{z}_1 \bar{z}_3 - z_3 \bar{z}_1 + \bar{z}_2 \bar{z}_3 z_3 - z_2 \bar{z}_2 \bar{z}_3 - z_2 \bar{z}_2 z_3 - z_3 \bar{z}_3 z_1 + \bar{z}_3 - z_3 \bar{z}_2 \bar{z}_1 + 2 \\
z_3 \bar{z}_2 \bar{z}_3 z_1 = (1 + z_3 \bar{z}_1)(-1 + z_3 \bar{z}_2)^2 \left( z_2 \bar{z}_3 - 1 \right)
\]  

(1.8)

Although the above expression is a rational function of \( z_1, z_2, \bar{z}_1, \bar{z}_2 \) and \( \bar{z}_3 \), when we equate it to the experimentally measured value and take it's real and imaginary parts, equations for two conic sections result. This is clear from examining the denominator and seeing that the unknown, \( z_3 \), occurs only with constant coefficients, or possibly a coefficient of \( \bar{z}_3 \).

1.4 **Simulation Results**

The main question to be addressed is the effect of noise on the above method, and so we need to know what is the nature of the noise that we expect from a panoramic sensor when measuring angles. From experiment we have found the noise in the device when measuring angles to have a standard deviation of about .05 degrees at a range of one meter.

![Figure 1.5](image)

There are many parameters that can be varied in simulation, such as the magnitude of the point being reconstructed, the number of data points, the numerical integration scheme and so on. Therefore we first fixed the two known points to be \( z_1 = 2i \) and \( z_2 = 2 + 2i \), fixed the number of measurements taken on the circle to be 5000, and considered the effect of zero mean Gaussian noise with \( \sigma \) ranging from 0 to .1 degree, and chose \( z_3 = 5, 7.5 \) and 10. (Note that our circle determines the scale - it has radius 1.) Given all of the above parameters 100 trials were performed and the error in each case was averaged and a percent error computed. We measured
error by computing the distance in the complex plane between the estimate and the true value, and then dividing by the magnitude of the true value. For each choice of $z_3$ this yields a plot of percent error versus the standard deviation in the noise, which can be seen in the left plot of figure 1.5.

The plot in the right of figure 1.5 illustrates the percent error in reconstruction versus the number of sample points used, i.e. the number of angular pairs measured. In this case we fixed $\sigma = .05$ considered three separate choices for $z_3$ to illustrate the effect of range. We see that in this plot there is rapid convergence to a constant percent error in each case.

### 1.5 Conclusion

In this paper we have introduced a new method for recovering the environmental lay-out from a monocular image sequence. To do this we have introduced certain spatiotemporal signatures that give rise to quadratic equations for the solution of the problem. These signatures are based on the principle of reconstruction from controlled motion where the precise motion is not known, but the shape of the motion is known. The method we propose is robust to noise and outliers and while taking into account global information.
Bibliography


