Well-Posedness of Two-Dimensional Hydroelastic Waves

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Abstract

A well-posedness theory for the initial value problem for hydroelastic waves in two spatial dimensions is presented. This problem, which arises in numerous applications, describes the evolution of a thin elastic membrane in a two-dimensional potential flow. We use a model for the elastic sheet that accounts for bending stresses and membrane tension, but which neglects the mass of the membrane. The analysis is based on a vortex sheet formulation, and following earlier analyses and numerical computations in 2D interfacial flow with surface tension, we use an angle-arclength representation of the problem. We prove short-time well-posedness in Sobolev spaces. The proof is based on energy estimates, and the main challenge is to find a definition of the energy and estimates on high-order nonlocal terms so that an a priori bound can be obtained.

1 Introduction

The hydroelastic problem describes the interaction between elastic bodies and hydrodynamic flow. We are interested in the particular version of this problem where an elastic sheet or membrane evolves in a potential flow. This problem is important in biology, medicine, and ocean engineering, and arises, for example, as a model for the dynamics of flapping flags [1], heart valves [28], ice sheets in the ocean [36], and very large floating structures [35]. A review which summarizes recent work on the analysis, numerical simulation, and applications of the hydroelastic problem is given by Korobkin, Parau and Vanden Broeck [29].

This paper presents an existence and uniqueness theory for the initial value problem for hydroelastic waves. Recently, Plotnikov and Toland [32] derived nonlinear equations that model the interaction of a thin, heavy elastic sheet with a three-dimensional inviscid, irrotational fluid. Their derivation is based on the Cosserat theory of shells satisfying Kirchoff's hypothesis, and accounts for bending stresses in the sheet as well as a membrane stretching tension. A similar model for the bending stress can be derived from minimization of the Willmore energy functional. Here, we consider a model for the 2D hydroelastic time-evolution problem (that is, a 1D interface evolving in 2D fluid flow) that is consistent with the hydroelastic formulation of Plotnikov and Toland, but which neglects the mass of the elastic membrane. The model is derived follwing the approach of [12]. Our main result is the local well-posedness of this problem.

The mathematical analysis of fluid-structure interaction presents significant challenges, and the only rigourous results for the hydroelastic problem that we are aware of are on the existence of steady

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traveling waves [33, 38, 39]. There is more work on the well-posedness of models for the interaction of a viscous (Navier-Stokes) fluid with an elastic membrane or solid. For example, Cheng et al. [14] have developed an existence and uniqueness theory for the problem of a nonlinear elastic 'bio-fluid' shell interacting with a viscous incompressible bulk fluid governed by the Navier-Stokes equations. Their model for the elastic shell involves a bending stress which extremizes the Willmore energy functional, similar to here, and a membrane or surface energy that is a function of the local area ratio. Wang et al. [40] prove the local well-posedness of an elastic surface model consisting of a viscous incompressible membrane fluid, but neglect the interaction with the bulk fluid. Their work generalizes an earlier analysis by Hu et al. [27]. Other work includes existence results for the interaction of a viscous fluid with, respectively, an elastic body moving in the fluid interior (e.g. [20]), finite thickness elastic shells [15], and regularized models of elastic plates (e.g. [13]). At first glance, one might surmise that theory developed in this paper can follow from the zero viscosity limit of [14]. A difficulty with this idea is that the Navier-Stokes and Euler's equations admit different kinds of boundary conditions, and the relationship between the zero viscosity limit of Navier-Stokes and Euler remains an open problem.

The approach developed in this paper relies on a boundary integral formulation of the hydroelastic problem. The analysis uses several important ideas from the numerical work of Hou, Lowengrub and Shelley (HLS) [25] and the analysis of Ambrose [3] of the initial value problem for vortex sheets with surface tension. These works recast the evolution equation by using the tangent angle θ and the arclength s as the dependent variables, rather than the natural Cartesian variables x and y. This choice of variables simplifies the curvature terms in the evolution equations. Additionally, they make a special choice of the tangential velocity $V(\alpha, t)$ of the interface (which may be chosen arbitrarily, and defines the parameterization α) so that s_α is independent of α. With this choice of V, the parameterization, normalized so that α is between 0 and 2π, is an equal arclength parameterization. There is a jump in velocity at the interface which is then a vortex sheet, and we denote the vortex sheet strength by γ. These choices simplify the analysis since the leading order or high derivative terms are linear as functions of θ and γ.

The analysis requires special care in the handling of terms with the highest number of spatial derivatives, which are contained within a singular integral operator known as the Birkhoff-Rott integral. To facilitate this, our proof makes use of the small scale decomposition (SSD) introduced in HLS. In the SSD, the leading order or highest derivative terms that are dominant at small spatial scales are identified and written in a simple form involving Hilbert transforms, rather than the more complicated Birkhoff-Rott integral. The small scale decomposition was used in HLS for computational purposes, but here it is employed in an essential way to simplify terms that need to be treated carefully in the analysis.

The main result of this paper is that the initial value problem for the hydroelastic flow of a periodic interface is well-posed in Sobolev spaces. In particular, given periodic initial data $\theta(\cdot,) \in H^s$ and $\gamma(\cdot, 0) \in H^{s-3/2}$ for s large enough (so that the interface variables $x(\alpha, t)$ and $y(\alpha, t)$ are in H^{s+1}) there is a nonzero time in which the solution exists, is unique, has the same regularity as the initial conditions, and depends continuously on the data.

The proof uses energy methods. The analysis is similar to that for vortex sheets with surface tension in [3], and the main challenge compared to the previous analysis is that the elasticity introduces higher order (nonlocal) terms, some of which are nonlinear. This requires a different definition of energy and more care in the energy estimates to achieve closure, that is, a bound on the time derivative of the energy by a function of the energy itself, which is a critical step in the proof.

The rest of this paper is organized as follows. In §2 we present an instructive example that illustrates the essential features of the energy estimate. Governing equations and preliminary estimates are presented in §3 and §4. The main existence proof is given in §5, and uniqueness and continuous dependence of the solution on the data is demonstrated in §6. Concluding remarks are given in §7. The appendix §8 derives an expression for the pressure jump at the interface that is used in the model.

1.1 Function spaces, norms, operators, and notation

Derivatives with respect to the independent variables t and α will be denoted either by using the partial derivative operators ∂_t and ∂_α, or with subscripts; thus $f_t = \partial_t f$, $f_\alpha = \partial_\alpha f$, $f_{\alpha\alpha} = \partial_\alpha^2 f$, and so on.

We comment now about the function spaces we will use. We use the L^2-based Sobolev spaces in the 2π-periodic setting. For $f \in H^k$ with $k \in \mathbb{N}$, we use the following as the norm:

$$\|f\|_k = \left(\int_0^{2\pi} f^2(\alpha) + (\partial_\alpha^k f(\alpha))^2 \; d\alpha \right)^{1/2}.$$

For $f \in H^{k+\frac{1}{2}}$, with $k \in \mathbb{N}$, we use the following as the norm:

$$\|f\|_{k+\frac{1}{2}} = \left(\int_0^{2\pi} f^2(\alpha) + (\partial_\alpha^k f(\alpha))(H\partial_\alpha^{k+1} f(\alpha)) \; d\alpha \right)^{1/2}.$$

Here, H is the periodic Hilbert transform, which has symbol $\hat{H}(\xi) = -i\mathrm{sgn}(\xi)$. (Notice that if f has mean zero, then $H^2 f = -f$. For more information on the periodic Hilbert transform, the interested reader might consult [24]). Using Plancherel's Theorem, it is clear that

$$\|f\|_{k+\frac{1}{2}} = \left(\sum_\xi (1 + |\xi|^{2k+1}) |\hat{f}(\xi)|^2 \right)^{1/2},$$

so this is equivalent to any other usual definition of the $H^{k+\frac{1}{2}}$ norm.

We will frequently use the notation Λ for the operator $\Lambda = H\partial_\alpha$; with this definition, the symbol of Λ is $\hat{\Lambda}(\xi) = |\xi|$, and this implies that Λ is self-adjoint. This implies the following, which we will use many times:

$$\frac{d}{dt} \int_0^{2\pi} g\Lambda g \; d\alpha = 2 \int_0^{2\pi} g\Lambda g_t \; d\alpha = 2 \int_0^{2\pi} g_t \Lambda g \; d\alpha. \tag{1}$$

This will be relevant as we estimate the growth of quantities which are equivalent to $H^{k+\frac{1}{2}}$ norms.

We will sometimes use the projection $\mathbb{P}$, which removes the zero mode of a periodic function:

$$\mathbb{P}f = f - \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) \; d\alpha.$$

We may sometimes denote the mean of a periodic function as $\langle\langle f \rangle\rangle$, so that we could say $\mathbb{P}f = f - \langle\langle f \rangle\rangle$. We also introduce the mean-zero antiderivative operator, ∂_α^{-1}. This is defined through its symbol as $\widehat{\partial_\alpha^{-1} f}(k) = \frac{1}{ik}\hat{f}(k)$, for $k \neq 0$, and $\widehat{\partial_\alpha^{-1} f}(0) = 0$.

2 An instructive example

Let c_1, c_2, c_3, c_4, and c_5 all be positive constants. We consider the following linear system, which has the same types of leading-order terms as the hydroelastic wave system we will be studying.

$$\theta_t = H(\gamma_\alpha) + (c_1 - c_2)\partial_\alpha^{-1} H(\gamma),$$

$$\gamma_t = -c_3\theta_{\alpha\alpha\alpha\alpha} + (c_4 - c_5)\theta_{\alpha\alpha}.$$

The coefficients of the second terms on the right-hand sides, which are $c_1 - c_2$ and $c_4 - c_5$, are written this way to make clear that we will be able to estimate these terms regardless of whether these coefficients are positive or negative. For the purpose of the present example, we take (θ, γ) to be a solution of this system which is sufficiently smooth for all of the integrals we are about to use to make sense.

The energy we will estimate will serve as an upper bound for a constant times the square of the H^3-norm of θ and the square of the $H^{3/2}$-norm of γ. We let $E(t)$ be given by

$$E(t) = E_0(t) + E_1(t) + E_2(t) + E_3(t) + E_4(t),$$

where

$$E_0(t) = \frac{1}{2}\int_0^{2\pi} \theta^2 + \gamma^2 \, d\alpha,$$

$$E_1(t) = \frac{c_3}{2}\int_0^{2\pi} (\partial_\alpha^3\theta)^2 \, d\alpha,$$

$$E_2(t) = \frac{1}{2}\int_0^{2\pi} (\partial_\alpha\gamma)(\Lambda\partial_\alpha\gamma) \, d\alpha,$$

and E_3 and E_4 will be defined shortly.

Taking the time derivative, it is straightforward that the growth of E_0 is bounded in terms of E:

$$\frac{dE_0}{dt} \le cE. \tag{2}$$

We next take the time derivative of E_1:

$$\frac{dE_1}{dt} = \int_0^{2\pi} (\partial_\alpha^3\theta)(\partial_\alpha^3\theta_t) \, d\alpha.$$

Substituting from the evolution equation for θ, this is

$$\frac{dE_1}{dt} = c_3\int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^4\gamma) \, d\alpha + c_3(c_1 - c_2)\int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^2\gamma) \, d\alpha. \tag{3}$$

Next, we take the time derivative of E_2:

$$\frac{dE_2}{dt} = \int_0^{2\pi} (\partial_\alpha\gamma)(\Lambda\partial_\alpha\gamma_t) \, d\alpha.$$

Plugging in from the evolution equation, this is

$$\frac{dE_2}{dt} = -c_3\int_0^{2\pi} (\partial_\alpha\gamma)(\Lambda\partial_\alpha^5\theta) \, d\alpha + (c_4 - c_5)\int_0^{2\pi} (\partial_\alpha\gamma)(\Lambda\partial_\alpha^3\theta) \, d\alpha.$$

We use the fact that Λ is self-adjoint, and we use the definition $\Lambda = H\partial_\alpha$:

$$\frac{dE_2}{dt} = -c_3 \int_0^{2\pi} (H\partial_\alpha^2\gamma)(\partial_\alpha^5\theta)\ d\alpha + (c_4 - c_5) \int_0^{2\pi} (H\partial_\alpha^2\gamma)(\partial_\alpha^3\theta)\ d\alpha.$$

We integrate by parts twice in the first integral on the right-hand side:

$$\frac{dE_2}{dt} = -c_3 \int_0^{2\pi} (H\partial_\alpha^4\gamma)(\partial_\alpha^3\theta)\ d\alpha + (c_4 - c_5) \int_0^{2\pi} (H\partial_\alpha^2\gamma)(\partial_\alpha^3\theta)\ d\alpha. \tag{4}$$

We add (3) and (4), finding that the terms with the most derivatives cancel. We are left with

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} = [(c_4 + c_1c_3) - (c_5 + c_2c_3)] \int_0^{2\pi} (H\partial_\alpha^2\gamma)(\partial_\alpha^3\theta)\ d\alpha. \tag{5}$$

Notice that this integral is not bounded in terms of the energy, since the energy controls three derivatives of θ and $3/2$ of a derivative of γ; thus, this integral contains terms with $1/2$ of a derivative more than we can control. We will cancel this by using E_3 and E_4, which we now define as

$$E_3(t) = \frac{d_1}{2} \int_0^{2\pi} (\partial_\alpha^2\theta)^2\ d\alpha,$$

$$E_4(t) = \frac{d_2}{2} \int_0^{2\pi} \gamma(\Lambda\gamma)\ d\alpha.$$

The positive constants d_1 and d_2 will be specified soon.

We take the time derivative of E_3 :

$$\frac{dE_3}{dt} = d_1 \int_0^{2\pi} (\partial_\alpha^2\theta)(\partial_\alpha^2\theta_t)\ d\alpha.$$

Plugging in from the evolution equation, this is

$$\frac{dE_3}{dt} = d_1 \int_0^{2\pi} (\partial_\alpha^2\theta)(H\partial_\alpha^3\gamma)\ d\alpha + d_1(c_1 - c_2) \int_0^{2\pi} (\partial_\alpha^2\theta)(H\partial_\alpha\gamma)\ d\alpha.$$

We integrate by parts once in the first integral on the right-hand side:

$$\frac{dE_3}{dt} = -d_1 \int_0^{2\pi} (\partial_\alpha^3\theta)(H\partial_\alpha^2\gamma)\ d\alpha + d_1(c_1 - c_2) \int_0^{2\pi} (\partial_\alpha^2\theta)(H\partial_\alpha\gamma)\ d\alpha. \tag{6}$$

We next take the time derivative of E_4 :

$$\frac{dE_4}{dt} = d_2 \int_0^{2\pi} \gamma(\Lambda\gamma_t)\ d\alpha.$$

We plug in from the evolution equation, finding the following:

$$\frac{dE_4}{dt} = -c_3d_2 \int_0^{2\pi} \gamma(\Lambda\partial_\alpha^4\theta)\ d\alpha + d_2(c_4 - c_5) \int_0^{2\pi} \gamma(\Lambda\partial_\alpha^2\theta)\ d\alpha.$$

For the first integral on the right-hand side, we use the fact that Λ is self-adjoint, and we use the definition $\Lambda = H\partial_\alpha$, and we also integrate by parts once (we also use $\Lambda = H\partial_\alpha$ in the second integral):

$$\frac{dE_4}{dt} = c_3d_2 \int_0^{2\pi} (H\partial_\alpha^2\gamma)(\partial_\alpha^3\theta)\ d\alpha + d_2(c_4 - c_5) \int_0^{2\pi} \gamma(H\partial_\alpha^3\theta)\ d\alpha. \tag{7}$$

We now add (5), (6), and (7), to find the following:

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} + \frac{dE_3}{dt} + \frac{dE_4}{dt} = \left[-d_1 + c_3 d_2 + c_4 + c_1 c_3 - c_5 - c_2 c_3\right] \int_0^{2\pi} (H\partial_\alpha^2 \gamma)(\partial_\alpha^3 \theta)\ d\alpha$$
$$+ d_1(c_1 - c_2) \int_0^{2\pi} (\partial_\alpha^2 \theta)(H\partial_\alpha \gamma)\ d\alpha + d_2(c_4 - c_5) \int_0^{2\pi} \gamma (H\partial_\alpha^3 \theta)\ d\alpha. \quad (8)$$

If we choose

$$d_1 = c_4 + c_1 c_3, \qquad d_2 = \frac{c_5 + c_2 c_3}{c_3},$$

then the first integral on the right-hand side of (8) vanishes. We then have

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} + \frac{dE_3}{dt} + \frac{dE_4}{dt} = d_1(c_1 - c_2) \int_0^{2\pi} (\partial_\alpha^2 \theta)(H\partial_\alpha \gamma)\ d\alpha + d_2(c_4 - c_5) \int_0^{2\pi} \gamma (H\partial_\alpha^3 \theta)\ d\alpha.$$

Since the remaining integrals involve at most one derivative of γ and at most three derivatives of θ, this can be estimated in terms of the energy:

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} + \frac{dE_3}{dt} + \frac{dE_4}{dt} \le cE. \quad (9)$$

Adding (2) and (9), we get

$$\frac{dE}{dt} = \frac{dE_0}{dt} + \frac{dE_1}{dt} + \frac{dE_2}{dt} + \frac{dE_3}{dt} + \frac{dE_4}{dt} \le cE.$$

This implies that the energy grows at most exponentially. Since the constants d_1, d_2, and c_3 are positive, we have the following:

$$\min\left\{\frac{1}{2}, \frac{c_3}{2}\right\} \left(\|\theta\|_3^2 + \|\gamma\|_{3/2}^2\right) \le E(t) \le E(0)e^{ct}.$$

Thus, the norm of (θ, γ) grows at most exponentially.

3 Equations of motion

In this section, we formulate the evolution equations for periodic hydroelastic waves, using the $\theta - L$ formulation of Hou, Lowengrub, and Shelley [25], [26]. This formulation has previously been used by the first author to develop the well-posedness theory of vortex sheets, water waves, and Hele-Shaw flows [3], [4], [5], [7]. Other authors have also used this formulation to prove results for water waves and Hele-Shaw flows including well-posedness, stability, and regularity results, among others [16], [17], [19], [21], [23], [42], [43].

We consider an interface S separating two inviscid, irrotational, incompressible fluids. The lower (respectively, upper) fluid is denoted by a subscript 1 (respectively, 2). The one-dimensional free surface is $(x(\alpha,t), y(\alpha,t))$, where α is the parameter along the curve, and t is time. We take the curve to be 2π-periodic, so that

$$x(\alpha + 2\pi, t) = x(\alpha, t) + 2\pi, \qquad y(\alpha + 2\pi, t) = y(\alpha, t),$$

for all α and t. We let $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ be the unit tangent and normal vectors along the curve, defined as

$$\hat{\mathbf{t}} = \frac{(x_\alpha, y_\alpha)}{s_\alpha}, \qquad \hat{\mathbf{n}} = \frac{(-y_\alpha, x_\alpha)}{s_\alpha},$$

with the arclength element, s_α defined by

$$s_\alpha^2 = x_\alpha^2 + y_\alpha^2.$$

We let U and V denote the normal and tangential velocities of the free surface, so that

$$(x, y)_t = U\hat{\mathbf{n}} + V\hat{\mathbf{t}}. \tag{10}$$

We introduce θ, the tangent angle that the curve forms with the horizontal, defined as $\theta = \tan^{-1}(y_\alpha/x_\alpha)$. We can infer evolution equations for s_α and θ from (10) [25]; we find

$$s_{\alpha,t} = V_\alpha - \theta_\alpha U, \tag{11}$$

$$\theta_t = \frac{U_\alpha + V\theta_\alpha}{s_\alpha}. \tag{12}$$

We make note of the following geometric identities:

$$\hat{\mathbf{t}}_\alpha = \theta_\alpha \cdot \hat{\mathbf{n}}, \qquad \hat{\mathbf{n}}_\alpha = -\theta_\alpha \cdot \hat{\mathbf{t}}. \tag{13}$$

Of course, we also have the relationship between curvature and tangent angle:

$$\kappa = \frac{\theta_\alpha}{s_\alpha}.$$

While the normal velocity is dictated by the physics of the problem, the tangential velocity is not. That is to say, changing the tangential velocity only changes the parameterization of the interface, and so we may use the tangential velocity to enforce our preferred parameterization. Our preferred parameterization is a normalized arclength parameterization: we would like s_α to be independent of α. If we let $L(t)$ denote the length of one period of the curve, then we would like $s_{\alpha,t}(\alpha, t) = L(t)/2\pi$ for all t. If this equation holds at the initial time, then it will hold at later times as long as

$$s_{\alpha,t} = \frac{L_t}{2\pi}.$$

Since $L(t) = \int_0^{2\pi} s_\alpha(\alpha, t)\ d\alpha$, we see from (11) using the periodicity of $V(\alpha, t)$ that

$$L_t = -\int_0^{2\pi} \theta_\alpha U\ d\alpha.$$

Considering again (11), this implies

$$V_\alpha = \frac{L_t}{2\pi} + \theta_\alpha U = \mathbb{P}(\theta_\alpha U). \tag{14}$$

Integrating, we thus have

$$V = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha U) + V(0, t). \tag{15}$$

The integration constant $V(0,t)$ is later chosen so that the mean of V is the same as the mean of $\mathbf{W}\cdot\hat{\mathbf{t}}$.

Since there is no vorticity in the bulk of the fluid, we are able to use a vortex sheet formulation. The average of the upper and lower fluid velocities evlauated at the interface S is denoted by $\mathbf{W} = (W_1, W_2)$ and is specified by the Birkhoff-Rott integral. In terms of the complex notation $(x,y) \to x+iy$, the Birkhoff-Rott integral is given by

$$W_1 - iW_2 = \frac{1}{4\pi i}\mathrm{PV}\int_0^{2\pi} \gamma(\alpha')\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\, d\alpha'. \tag{16}$$

The curve z is the complex form of the interface (x,y) :

$$z(\alpha,t) = x(\alpha,t) + iy(\alpha,t).$$

The normal velocity is the normal component of the Birkhoff-Rott integral:

$$U = \mathbf{W}\cdot\hat{\mathbf{n}}. \tag{17}$$

The function γ is the vortex sheet strength; it is the jump in velocity (lower minus upper fluid) across the interface. Since the velocity potential on each side of the interface satisfies a Bernoulli equation, upon taking the limit at the interface, an evolution equation for the jump in potential across the interface can be found. Differentiating this leads to the following evolution equation for γ :

$$\gamma_t = -\frac{2}{\rho_1+\rho_2}[p]_\alpha + \frac{2\pi}{L}((V-\mathbf{W}\cdot\hat{\mathbf{t}})\gamma)_\alpha - 2A\left(\frac{L}{2\pi}\mathbf{W}_t\cdot\hat{\mathbf{t}} + \frac{\pi^2}{L^2}\gamma\gamma_\alpha - (V-\mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} + gy_\alpha\right). \tag{18}$$

where $[p] = (p_1 - p_2)\|_S$ is the jump in pressure at the interface and g is the acceleration due to gravity. For details of the derivation of (18), the reader could consult [7] or [10] . The jump in pressure across the interface is given by (see §8)

$$[p] = E_b(\kappa_{ss} + \frac{\kappa^3}{2} - c(t)\kappa) \tag{19}$$

In the above, ρ_1 and ρ_2 are the densities of fluid 1 and fluid 2, respectively. The Atwood number, A, is $A = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}$, and E_b is the bending modulus.

We present a nondimensionalized version of the equation for γ. Lengths are nondimensionalized by a representative length of the periodic domain l, pressure is made dimensionless by E_B/l^3, the surface tension parameter $c(t)$ by E_B/l^2, velocity by $\sqrt{lg}$, γ by $l\sqrt{g}$, and time by $\sqrt{l/g}$. Introduce the dimensionless parameter

$$S = \frac{E_B}{(\rho_1+\rho_2)lg}.$$

The nondimensional equation for γ is then

$$\gamma_t = S\left(-\frac{\kappa_{\alpha\alpha}}{s_\alpha^2} - \frac{\kappa^3}{2} + \bar{c}_1\kappa\right)_\alpha + \frac{((V-\mathbf{W}\cdot\hat{\mathbf{t}})\gamma)_\alpha}{s_\alpha}$$
$$- 2A\left[(\mathbf{W}_t\cdot\hat{\mathbf{t}})s_\alpha - (V-\mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} + \frac{1}{8}\partial_\alpha\left(\frac{\gamma^2}{s_\alpha^2}\right) + y_\alpha\right].$$

If we use the relationship $\kappa = \theta_\alpha / s_\alpha$, and distribute the derivative on the first term on the right-hand side, this is

$$\gamma_t = S\left(-\frac{\partial_\alpha^4 \theta}{s_\alpha^3} - \frac{3\theta_\alpha^2 \theta_{\alpha\alpha}}{2s_\alpha^3} + \bar{c}_1 \frac{\theta_{\alpha\alpha}}{s_\alpha}\right) + \frac{((V - \mathbf{W}\cdot\hat{\mathbf{t}})\gamma)_\alpha}{s_\alpha} - 2A\left[(\mathbf{W}_t \cdot \hat{\mathbf{t}})s_\alpha - (V - \mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha \cdot \hat{\mathbf{t}} + \frac{1}{8}\partial_\alpha\left(\frac{\gamma^2}{s_\alpha^2}\right) + y_\alpha\right]. \quad (20)$$

3.1 Approximating the Birkhoff-Rott integral

We introduce a bit of notation which is helpful to us as we switch back forth between real and complex notation. We let $\Phi : \mathbb{R}^2 \to \mathbb{C}$ be the complexification map,

$$\Phi(a, b) = a + ib.$$

Thus, for instance, $z = \Phi(x, y)$. We will denote the complex conjugate with *, as in $z^* = x - iy$.

We introduced previously the periodic Hilbert transform, H, but we only discussed it in terms of its symbol. There is an integral form of the Hilbert transform; if $f \in L^2$, say, then

$$Hf(\alpha) = \frac{1}{2\pi}\mathrm{PV}\int_0^{2\pi} f(\alpha')\cot\left(\frac{1}{2}(\alpha - \alpha')\right)\, d\alpha'.$$

We notice that (16), the formula for the Birkhoff-Rott integral, looks something like the Hilbert transform. We thus introduce the following operator, which is the error in approximating an integral like (16) with a Hilbert transform:

$$\mathcal{K}[z_d]f(\alpha) = \frac{1}{4\pi i}\int_0^{2\pi} f(\alpha')\left[\cot\left(\frac{1}{2}(z_d(\alpha) - z_d(\alpha'))\right) - \frac{1}{z_\alpha(\alpha')}\cot\left(\frac{1}{2}(\alpha - \alpha')\right)\right] d\alpha'. \quad (21)$$

Here, we have introduced the quantity z_d, which is defined as

$$z_d(\alpha, t) = z(\alpha, t) - z(0, t);$$

we have already used in (21) the fact that $z(\alpha, t) - z(\alpha', t) = z_d(\alpha, t) - z_d(\alpha', t)$. It is convenient to use z_d instead of z because z_d is determined uniquely from θ, while z is not. Notice furthermore that $\partial_\alpha z = \partial_\alpha z_d$. We will also need the commutator of the Hilbert transform with multiplication by a smooth function:

$$[H, \phi]f(\alpha) = H(\phi f)(\alpha) - \phi(\alpha)H(f)(\alpha).$$

The operators $\mathcal{K}[z_d]$ and $[H, \phi]$ are both smoothing operators; estimates demonstrating this smoothing will be given in Section 4 below.

We will not provide the full details here, but having introduced these operators, we are able to write $\mathbf{W}_\alpha$ as follows:

$$\mathbf{W}_\alpha = \frac{\pi}{L}H(\gamma_\alpha)\hat{\mathbf{n}} - \frac{\pi}{L}H(\gamma\theta_\alpha)\hat{\mathbf{t}} + \mathbf{m}, \quad (22)$$

where $\mathbf{m}$ is a collection of smoother terms given by

$$\Phi(\mathbf{m})^* = z_\alpha \mathcal{K}[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right). \quad (23)$$

Formulas (22) and (23) were initially developed by the first author in [3], and used subsequently in several papers, including most recently [5].

We now give a useful formula for $V - \mathbf{W}\cdot\hat{\mathbf{t}}$. Notice that we can use (13) to find that $(\mathbf{W}\cdot\hat{\mathbf{t}})_\alpha = \mathbf{W}_\alpha\cdot\hat{\mathbf{t}} + (\mathbf{W}\cdot\hat{\mathbf{n}})\theta_\alpha$. Since $V_\alpha = \frac{L_t}{2\pi} + \theta_\alpha U$, and since $U = \mathbf{W}\cdot\hat{\mathbf{n}}$, we see that

$$(V - \mathbf{W}\cdot\hat{\mathbf{t}})_\alpha = -\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} + \frac{L_t}{2\pi}.$$

We substitute from (22) to find

$$(V - \mathbf{W}\cdot\hat{\mathbf{t}})_\alpha = \frac{\pi}{L}H(\gamma\theta_\alpha) - \mathbf{m}\cdot\hat{\mathbf{t}} + \frac{L_t}{2\pi}.$$

This can be rewritten by using the operator $\mathbb{P}$; note that the left-hand side has no mean, and that a Hilbert transform has no mean. Thus, we have the following:

$$(V - \mathbf{W}\cdot\hat{\mathbf{t}})_\alpha = \frac{\pi}{L}H(\gamma\theta_\alpha) - \mathbb{P}(\mathbf{m}\cdot\hat{\mathbf{t}}).$$

We apply the operator ∂_α^{-1}, and we also introduce the notation $V_W = V - \mathbf{W}\cdot\hat{\mathbf{t}}$: get

$$V_W = \partial_\alpha^{-1}\left(\frac{\pi}{L}H(\gamma\theta_\alpha) - \mathbb{P}(\mathbf{m}\cdot\hat{\mathbf{t}})\right). \tag{24}$$

We note that it is implicit in this that the mean of V is chosen to be the same as the mean of $\mathbf{W}\cdot\hat{\mathbf{t}}$; this is possible since the equation defining V is an equation for V_α, and the mean of V is then free to be chosen.

We close this section with an expression for θ_t. We start from (12), and we use the equation $U = \mathbf{W}\cdot\hat{\mathbf{n}}$ to find

$$U_\alpha = \mathbf{W}_\alpha\cdot\hat{\mathbf{n}} - (\mathbf{W}\cdot\hat{\mathbf{t}})\theta_\alpha.$$

Together with (22), this implies

$$\theta_t = \frac{2\pi^2}{L^2}H(\gamma_\alpha) + \frac{2\pi}{L}V_W\theta_\alpha + \frac{2\pi}{L}\mathbf{m}\cdot\hat{\mathbf{n}}. \tag{25}$$

3.2 Calculation of $(\mathbf{W}_t\cdot\hat{\mathbf{t}})s_\alpha$

In this section, we will rewrite $(\mathbf{W}_t\cdot\hat{\mathbf{t}})s_\alpha$, which appears on the right-hand side of (20). To begin, we can write $\mathbf{W}_t$ as

$$\Phi(\mathbf{W}_t^*) = \frac{1}{4\pi i}\mathrm{PV}\int \gamma_t(\alpha')\cot\left(\frac{1}{2}(z(\alpha) - z(\alpha'))\right)\,d\alpha' \\ - \frac{1}{8\pi i}\mathrm{PV}\int \gamma(\alpha')(z_t(\alpha) - z_t(\alpha'))\csc^2\left(\frac{1}{2}(z(\alpha) - z(\alpha'))\right)\,d\alpha'.$$

We then write $(\mathbf{W}_t\cdot\hat{\mathbf{t}})s_\alpha$ as

$$(\mathbf{W}_t\cdot\hat{\mathbf{t}})s_\alpha = \mathcal{J}[z_d](\gamma_t) + R,$$

with the operator $\mathcal{J}[z_d]$ defined by

$$\mathcal{J}[z_d](f)(\alpha) = \mathrm{Re}\left\{\frac{z_\alpha}{4\pi i}\mathrm{PV}\int f(\alpha')\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\,d\alpha'\right\},$$

and the term R is given by

$$R = \mathrm{Re}\left\{-\frac{z_\alpha}{8\pi i}\mathrm{PV}\int \gamma(\alpha')(z_t(\alpha)-z_t(\alpha'))\csc^2\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\,d\alpha'\right\}.$$

We will continue to rewrite R. To begin, we put in some factors of $z_\alpha(\alpha')$ and recognize a perfect derivative:

$$R = \mathrm{Re}\left\{-\frac{z_\alpha}{8\pi i}\mathrm{PV}\int \frac{\gamma(\alpha')}{z_\alpha(\alpha')}(z_t(\alpha)-z_t(\alpha'))z_\alpha(\alpha')\csc^2\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\,d\alpha'\right\}$$
$$= \mathrm{Re}\left\{-\frac{z_\alpha}{4\pi i}\mathrm{PV}\int \frac{\gamma(\alpha')}{z_\alpha(\alpha')}(z_t(\alpha)-z_t(\alpha'))\partial_{\alpha'}\left(\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\right)\,d\alpha'\right\}.$$

We then integrate by parts:

$$R = \mathrm{Re}\left\{\frac{z_\alpha}{4\pi i}\mathrm{PV}\int \partial_{\alpha'}\left(\frac{\gamma(\alpha')(z_t(\alpha)-z_t(\alpha'))}{z_\alpha(\alpha')}\right)\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\,d\alpha'\right\}$$

We apply the derivative, and write $R = R_1+R_2$, where R_1 and R_2 are given by the following formulas:

$$R_1 = \mathrm{Re}\left\{-\frac{z_\alpha}{4\pi i}\mathrm{PV}\int \frac{\gamma(\alpha')z_{t\alpha}(\alpha')}{z_\alpha(\alpha')}\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\,d\alpha'\right\},$$

$$R_2 = \mathrm{Re}\left\{\frac{z_\alpha}{4\pi i}\mathrm{PV}\int \partial_{\alpha'}\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\right)(z_t(\alpha)-z_t(\alpha'))\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\,d\alpha'\right\}.$$

Of these, we need to continue to rewrite R_1, since it has significant terms we need to treat carefully. For R_2, we need to rewrite it in order to see that it contains no such significant terms. We treat R_2 first, by using $\mathcal{K}$:

$$R_2 = -\mathrm{Re}\left\{\frac{z_\alpha}{2i}[H,z_t]\left(\frac{1}{z_\alpha}\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\} + \mathrm{Re}\left\{z_\alpha z_t\mathcal{K}[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\} - \mathrm{Re}\left\{z_\alpha\mathcal{K}[z_d]\left(z_t\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\}. \quad (26)$$

We add and subtract in R_1:

$$R_1 = \mathrm{Re}\left\{-\frac{z_\alpha}{2i}H\left(\frac{\gamma z_{t\alpha}}{z_\alpha^2}\right)\right\} - \mathrm{Re}\left\{z_\alpha\mathcal{K}[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\}.$$

To extract the important terms from this, we need a useful expression for $z_{t\alpha}$. We know that $z_\alpha = s_\alpha e^{i\theta}$ and $s_\alpha = L/2\pi$, so we have

$$z_{t\alpha} = \frac{L_t}{2\pi}e^{i\theta} + i\theta_t z_\alpha = \frac{L_t}{L}z_\alpha + i\theta_t z_\alpha.$$

Using this, we have the following for R_1 :

$$R_1 = \mathrm{Re}\left\{-\frac{z_\alpha}{2}H\left(\frac{\gamma\theta_t}{z_\alpha}\right)\right\} + \mathrm{Re}\left\{-\frac{z_\alpha L_t}{2Li}H\left(\frac{\gamma}{z_\alpha}\right)\right\} - \mathrm{Re}\left\{z_\alpha\mathcal{K}[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\}.$$

To continue, we substitute for θ_t from (25):

$$R_1 = \mathrm{Re}\left\{-\frac{\pi^2 z_\alpha}{L^2}H\left(\frac{\gamma H(\gamma_\alpha)}{z_\alpha}\right)\right\} + \mathrm{Re}\left\{-\frac{\pi z_\alpha}{L}H\left(\frac{\gamma V_W\theta_\alpha}{z_\alpha}\right)\right\} + \mathrm{Re}\left\{-\frac{\pi z_\alpha}{L}H\left(\frac{\gamma\mathbf{m}\cdot\hat{\mathbf{n}}}{z_\alpha}\right)\right\}$$
$$+ \mathrm{Re}\left\{-\frac{z_\alpha L_t}{2Li}H\left(\frac{\gamma}{z_\alpha}\right)\right\} - \mathrm{Re}\left\{z_\alpha\mathcal{K}[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\}. \quad (27)$$

Next, for the first two terms on the right-hand side of (27), we pull some things through the Hilbert transform, incurring commutators in the process:

$$R_1 = \frac{\pi^2}{L^2}\gamma\gamma_\alpha - \frac{\pi}{L}V_W H(\gamma\theta_\alpha) + \mathrm{Re}\left\{-\frac{\pi^2 z_\alpha}{L^2}\left[H, \frac{\gamma}{z_\alpha}\right](H(\gamma_\alpha))\right\} + \mathrm{Re}\left\{-\frac{\pi z_\alpha}{L}\left[H, \frac{V_W}{z_\alpha}\right](\gamma\theta_\alpha)\right\}$$
$$+ \mathrm{Re}\left\{-\frac{\pi z_\alpha}{L}H\left(\frac{\gamma\mathbf{m}\cdot\hat{\mathbf{n}}}{z_\alpha}\right)\right\} + \mathrm{Re}\left\{-\frac{z_\alpha L_t}{2Li}H\left(\frac{\gamma}{z_\alpha}\right)\right\} - \mathrm{Re}\left\{z_\alpha\mathcal{K}[z_d]\left(\frac{\gamma z_{t\alpha}}{z_\alpha}\right)\right\}. \quad (28)$$

We give the name R_3 to the sum of the last four terms on the right-hand side of (28), so that we have

$$R_1 = \frac{\pi^2}{L^2}\gamma\gamma_\alpha - \frac{\pi}{L}V_W H(\gamma\theta_\alpha) + R_3. \quad (29)$$

The conclusion of this subsection is the following formula:

$$(\mathbf{W}_t\cdot\hat{\mathbf{t}})s_\alpha = -\frac{\pi}{L}V_W H(\gamma\theta_\alpha) + \frac{\pi^2}{L^2}\gamma\gamma_\alpha + \mathcal{J}[z_d]\gamma_t + R_2 + R_3. \quad (30)$$

We will estimate $\mathcal{J}[z_d](\gamma_t)$, R_2, and R_3 in Section 4 below.

3.3 Our small-scale decomposition

We are now going to rewrite the above evolution equations in an important ways: we will emphasize the terms in the θ_t and γ_t evolution equations which must be treated carefully in the energy estimates.

We pick up from (20), and we replace s_α with $L/2\pi$. We also introduce the notation

$$\widetilde{S}(t) = \frac{S}{s_\alpha^3} = \frac{8\pi^3 S}{L^3}.$$

We also apply the α−derivative in the second term on the right hand side in (20). These considerations yield the following:

$$\gamma_t = \widetilde{S}\left(-\partial_\alpha^4\theta - \theta_{\alpha\alpha}\left(\frac{3}{2}\theta_\alpha^2 - \frac{L^2\bar{c}_1}{4\pi^2}\right)\right) + \frac{2\pi\gamma\partial_\alpha V_W}{L} + \frac{2\pi V_W\gamma_\alpha}{L}$$
$$- 2A\left[(\mathbf{W}_t\cdot\hat{\mathbf{t}})s_\alpha - V_W\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} + \frac{1}{8}\partial_\alpha\left(\frac{4\pi^2\gamma^2}{L^2}\right) - y_\alpha\right].$$

Next, we substitute for both $(\mathbf{W}_t \cdot \hat{\mathbf{t}})s_\alpha$ from (30) and for $\mathbf{W}_\alpha \cdot \hat{\mathbf{t}}$, using (22). In doing so, there is an important cancellation, since both of these terms contribute a term $\frac{\pi}{L} V_W H(\gamma\theta_\alpha)$, with opposite signs. After making this cancellation, we are left with

$$\gamma_t = \widetilde{S}\left(-\partial_\alpha^4\theta - \theta_{\alpha\alpha}\left(\frac{3}{2}\theta_\alpha^2 - \frac{L^2\bar{c}_1}{4\pi^2}\right)\right) + \frac{2\pi\gamma\partial_\alpha V_W}{L} + \frac{2\pi V_W\gamma_\alpha}{L} - 2A\left[\mathcal{J}[z_d]\gamma_t + R_2 + R_3 - V_W\mathbf{m}\cdot\hat{\mathbf{t}} + \frac{\pi^2}{L^2}\gamma\gamma_\alpha - y_\alpha\right]$$

which is an integral equation for γ_t. Finally, we group together some like terms, writing the integral equation above as

$$\gamma_t = -\widetilde{S}\partial_\alpha^4\theta - \widetilde{S}\theta_{\alpha\alpha}Q_1 + \gamma_\alpha Q_2 + Q_3, \tag{31}$$

where Q_1, Q_2, and Q_3 are the following collections of smoother terms:

$$Q_1 := Q_1(\alpha,t) = \frac{3}{2}\theta_\alpha^2 - \frac{L^2\bar{c}_1}{4\pi^2},$$

$$Q_2 := Q_2(\alpha,t) = \frac{2\pi V_W}{L} - \frac{2A\pi^2}{L^2}\gamma,$$

$$Q_3 := Q_3(\alpha,t) = \frac{2\pi\gamma\partial_\alpha V_W}{L} - 2A\mathcal{J}[z_d]\gamma_t - 2AR_2 - 2AR_3 + 2AV_W\mathbf{m}\cdot\hat{\mathbf{t}} - 2Ay_\alpha.$$

It will be helpful to have a brief notation for the evolution equations, so we introduce the following:

$$(\theta,\gamma)_t = \mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2). \tag{32}$$

The definitions of $\mathcal{B}_1$ and $\mathcal{B}_2$ are clearly just given by the right-hand sides of the the equations (25) and (31).

4 Preliminary estimates

We present without proof several important lemmas that are used in the subsequent estimates. The proofs of Lemmas 1 through 5 can be found in [3]. A version of Lemma 6 also appears in [3], but the particular form of the statement and proof of Lemma 6 is from [18] (see also [10] and [41]). Versions of many of these lemmas may also be found in [6] or [11].

The first lemma is a standard interpolation lemma for Sobolev spaces:

Lemma 1 *Let $m \geq 0$ and $\ell \geq m$ be given. Let $f \in H^\ell$ be given. Then, the following inequality holds:*

$$\|f\|_m \leq c\|f\|_\ell^{m/\ell}\|f\|_0^{1-m/\ell}. \tag{33}$$

The remainder operator $\mathcal{K}$ is a smoothing operator; the following lemma makes this precise. Establishing this requires an assumption on the chord arc quantity

$$q_1(\alpha,\alpha') = \frac{z_d(\alpha) - z_d(\alpha')}{\alpha - \alpha'}$$

which states that the interfacial curve z_d is not close to self-intersecting.

Lemma 2 *Let $n \geq 2$ be an integer. Assume $z_d \in H^n$. Assume there exists $\beta_1 > 0$ such that for all α and α',*

$$|q_1(\alpha, \alpha')| > \beta_1.$$

Then $\mathcal{K}[z_d] : H^1 \to H^{n-1}$ and $\mathcal{K}[z_d] : H^0 \to H^{n-2}$, with the estimates

$$\|\mathcal{K}[z_d]f\|_{n-1} \leq C_1 \|f\|_1 \exp\{C_2\|z_d\|_n\},$$

$$\|\mathcal{K}[z_d]f\|_{n-2} \leq C_1 \|f\|_0 \exp\{C_2\|z_d\|_n\}.$$

We also will need a Lipschitz estimate for $\mathcal{K}$, when we establish uniqueness and continuous dependence of solutions.

Lemma 3 *Let θ and θ' be in H^3. Let L and L' be the corresponding lengths of the associated curves z_d and z_d', and let q_1 and q_1' be the associated chord-arc quantities. Assume there exists positive constants β_1 and β_2 such that $L < \beta_2$ and $L' < \beta_2$, and for all α and α',*

$$|q_1(\alpha, \alpha')| > \beta_1, \qquad |q_1'(\alpha, \alpha')| > \beta_1.$$

Then the following Lipschitz estimate holds, for any $f \in H^3$:

$$\|\mathcal{K}[z_d]f - \mathcal{K}[z_d']f\|_3 \leq c\|\theta - \theta'\|_3 \|f\|_3.$$

A version of Lemma 3 was proved in [2] giving an estimate in H^1 rather than H^3. The same proof goes through, however, in H^3, so we omit it.

We have two different commutator estimates for the commutator of the Hilbert transform and multiplication by a smooth function. The first of these yields less regularity for the commutator, but requires less regularity on the functions.

Lemma 4 *Let $n \geq 1$ be an integer. Let $\phi \in H^n$ be given. Then $[H, \phi] : H^0 \to H^{n-1}$ and $[H, \phi] : H^{-1} \to H^{n-2}$, with the estimates*

$$\|[H, \phi]f\|_{n-1} \leq c\|\phi\|_n \|f\|_0,$$

$$\|[H, \phi]f\|_{n-2} \leq c\|\phi\|_n \|f\|_{-1}. \tag{34}$$

Our second commutator lemma gives higher regularity of the commutator, by requiring more regularity on the functions.

Lemma 5 *Let $j \geq 1$ be an integer. Let $n \geq 2j$ be an integer. Let $\phi \in H^n$ be given. Then, $[H, \phi] : H^{n-j} \to H^n$, with the estimate*

$$\|[H, \phi]f\|_n \leq c\|\phi\|_n \|f\|_{n-j}.$$

We need a lemma giving solvability of our γ_t integral equation (31).

Lemma 6 *Assume $z_d \in H^n$ for $n \geq 3$. The operator $(I + 2A\mathcal{J}[z_d])^{-1}$ is bounded from H^0 to H^0, with the estimate*

$$\left\|(I + 2A\mathcal{J}[z_d])^{-1} F\right\|_0 \leq c_1 \exp\{c_2\|z_d\|_3\}\|F\|_0.$$

5 Existence

Before proving existence of solutions, we first must introduce a regularized system of evolution equations. We will first prove existence of solutions for the regularized system, and then prove energy estimates for the regularized system. We will then be able to pass to the limit as the regularization parameter vanishes, finding that solutions of the non-regularized system of evolution equations exist.

5.1 The mollified system

We will need to be careful about reconstructing a curve from a tangent angle; this is because not every periodic tangent angle function will lead to a periodic curve. In particular, say η is our tangent angle function, perhaps at a step of an iteration procedure (so that η cannot be assumed to be a solution of our evolution equation). First we concern ourselves with defining the length of the curve; this comes from the horizontal periodicity.

The derivative of the horizontal component of the curve to be constructed from η is

$$x_\alpha[\eta, L] = \frac{L}{2\pi}\cos(\eta). \tag{35}$$

The horizontal periodicity requires that $x[\eta, L](2\pi) - x[\eta, L](0) = 2\pi$, so we have

$$2\pi = \int_0^{2\pi} x_\alpha[\eta, L](\alpha)\ d\alpha = \frac{L}{2\pi}\int_0^{2\pi} \cos(\eta(\alpha))\ d\alpha.$$

Solving for L, this is

$$L[\eta] = \frac{4\pi^2}{\int_0^{2\pi}\cos(\eta(\alpha))\ d\alpha}.$$

For $y_\alpha[\eta, L]$, we want $\int_0^{2\pi} y_\alpha[\eta, L](\alpha)\ d\alpha = 0$. To enforce our periodicity condition on $x[\eta, L]$, we were able to choose $L[\eta]$ accordingly; there is no corresponding choice we can make in this case. Instead, we simply must project the mean away. We define $y_\alpha[\eta, L]$ to be

$$y_\alpha[\eta] = \frac{L[\eta]}{2\pi}\mathbb{P}\sin(\eta), \tag{36}$$

where $\mathbb{P}$ is the projection which zeros out the mean.

We must define the mollified curve, and we use the above discussion as guidance. We let θ^ε be given. We let the length, L^ε, be defined as

$$L^\varepsilon = L[\theta^\varepsilon].$$

Naturally, since $\widetilde{S} = \frac{8\pi^3 S}{L^3}$, we will use the notation $\widetilde{S}^\varepsilon$:

$$\widetilde{S}^\varepsilon = \frac{8\pi^3 S}{(L^\varepsilon)^3}.$$

The derivative of the curve is given by

$$x_\alpha^\varepsilon = x_\alpha[\theta^\varepsilon, L^\varepsilon] = \frac{L^\varepsilon}{2\pi}\cos(\theta^\varepsilon), \tag{37}$$

$$y_\alpha^\varepsilon = y_\alpha[\theta^\varepsilon, L^\varepsilon] = \frac{L^\varepsilon}{2\pi}\mathbb{P}\sin(\theta^\varepsilon). \tag{38}$$

The mollified curve is then defined by integrating:

$$z_d^\varepsilon = \frac{L^\varepsilon}{2\pi}\int_0^\alpha \cos(\theta^\varepsilon) + i\mathbb{P}\sin(\theta^\varepsilon)\ d\alpha, \tag{39}$$

and the unit normal and tangent curves are defined to be

$$\hat{\mathbf{t}}^\varepsilon = (\cos(\theta^\varepsilon), \sin(\theta^\varepsilon)), \qquad \hat{\mathbf{n}}^\varepsilon = (-\sin(\theta^\varepsilon), \cos(\theta^\varepsilon)). \tag{40}$$

(Note that if θ^ε does not satisfy $\mathbb{P}\sin(\theta^\varepsilon) = \sin(\theta^\varepsilon)$, then these vectors $\hat{\mathbf{t}}^\varepsilon$ and $\hat{\mathbf{n}}^\varepsilon$ are not actually the unit tangent and normal vectors to the curve z_d^ε. We will ensure that when θ^ε is a solution of of the evolution equation (to be defined), that the property $\mathbb{P}\sin(\theta^\varepsilon) = \sin(\theta^\varepsilon)$ holds.)

For the exact evolution equations, we have $\theta_t = \frac{U_\alpha + V\theta_\alpha}{s_\alpha}$. If we study $\int_0^{2\pi}\sin(\theta(\alpha,t))\ d\alpha$ in this case, we find

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta(\alpha,t))\ d\alpha = \frac{1}{s_\alpha}\int_0^{2\pi}\cos(\theta)(U_\alpha + V\theta_\alpha)\ d\alpha.$$

We notice that $\cos(\theta)\theta_\alpha = \partial_\alpha\sin(\theta)$, and we integrate by parts for this term. We also recall (14), which says that $V_\alpha = \frac{L_t}{2\pi} + \theta_\alpha U$. These considerations yield the following:

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta(\alpha,t))\ d\alpha = \frac{1}{s_\alpha}\int_0^{2\pi}\left[\cos(\theta)U_\alpha - \sin(\theta)\theta_\alpha U\right]\ d\alpha - \frac{L_t}{L}\int_0^{2\pi}\sin(\theta)\ d\alpha.$$

Integrating by parts in the first term, we see that

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta(\alpha,t))\ d\alpha = -\frac{L_t}{L}\int_0^{2\pi}\sin(\theta)\ d\alpha.$$

This implies that the mean of $\sin(\theta)$ grows or decays exponentially, with the exponential growth rate related to L and L_t. Thus, for a solution, θ, of the exact evolution equations, we see that if the mean of $\sin(\theta)$ is initially zero, then it will remain zero at positive times.

We introduce the following analogue of (32) for the mollified system:

$$(\theta^\varepsilon, \gamma^\varepsilon)_t = (\mathcal{B}_1^\varepsilon + \mu^\varepsilon, \mathcal{B}_2^\varepsilon), \tag{41}$$

where we must now define $\mathcal{B}_1^\varepsilon$, $\mathcal{B}_2^\varepsilon$, and μ^ε. Of these, $\mathcal{B}_1^\varepsilon$ and $\mathcal{B}_2^\varepsilon$ will be rather clearly similar to $\mathcal{B}_1$ and $\mathcal{B}_2$, but with mollification operators applied in a variety of places. The other term, μ^ε, will be used to enforce our periodicity requirement, that $\mathbb{P}(\sin(\theta^\varepsilon)) = \sin(\theta^\varepsilon)$. We remark that μ^ε is taken to be a function of t only, and to not depend on α; that is, μ^ε will be related to spatial averages of other quantities, and will thus be a constant function with respect to the spatial variable.

The previous argument, for the non-mollified system, showed that the structure of the non-mollified evolution equations implies $\frac{d}{dt}\int_0^{2\pi}\sin(\theta(\alpha,t))\ d\alpha = 0$. For solutions of the mollified system, we no longer have the exact structure that we used previously, and we instead define μ^ε to achieve our desired goal. In particular, we have the following:

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta^\varepsilon)\ d\alpha = \int_0^{2\pi}\theta_t^\varepsilon\cos(\theta^\varepsilon)\ d\alpha = \int_0^{2\pi}\mathcal{B}_1^\varepsilon\cos(\theta^\varepsilon)\ d\alpha + \mu^\varepsilon\int_0^{2\pi}\cos(\theta^\varepsilon)\ d\alpha.$$

Since we want this to equal zero, we make the following definition of μ^ε :

$$\mu^\varepsilon = -\frac{\int_0^{2\pi} \mathcal{B}_1^\varepsilon \cos(\theta^\varepsilon)\ d\alpha}{\int_0^{2\pi} \cos(\theta^\varepsilon)\ d\alpha} = -\frac{L^\varepsilon}{4\pi^2}\int_0^{2\pi} \mathcal{B}_1^\varepsilon \cos(\theta^\varepsilon)\ d\alpha. \tag{42}$$

The point of this is that, if $(\theta^\varepsilon, \gamma^\varepsilon)$ solves (41), then $\mathbb{P}(\sin(\theta^\varepsilon)) = \sin(\theta^\varepsilon)$. In light of (35) and (36), this implies

$$|z_\alpha^\varepsilon(\alpha, t)| = \frac{L^\varepsilon(t)}{2\pi}, \quad \forall \alpha,$$

as desired; this would not be the case if the mean of $\sin(\theta^\varepsilon)$ were nonzero.

We let the mollifier with parameter ε be denoted χ_ε; this operator acts through truncation of the Fourier series, zeroing out modes with wavenumber larger than $1/\varepsilon$. As such, χ_ε is a projection, so that $\chi_\varepsilon^2 = \chi_\varepsilon$.

We now define $\mathcal{B}_1^\varepsilon$ and $\mathcal{B}_2^\varepsilon$. To begin, we make the following definitions:

$$\mathcal{B}_1^\varepsilon = \frac{2\pi^2}{(L^\varepsilon)^2}\chi_\varepsilon H(\gamma_\alpha^\varepsilon) + \frac{2\pi}{L^\varepsilon}\chi_\varepsilon\left(V_W^\varepsilon(\chi_\varepsilon\theta_\alpha^\varepsilon)\right) + \frac{2\pi}{L^\varepsilon}\mathbf{m}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon, \tag{43}$$

$$\mathcal{B}_2^\varepsilon = -\widetilde{S}^\varepsilon\chi_\varepsilon\partial_\alpha^4\theta^\varepsilon - \widetilde{S}^\varepsilon\chi_\varepsilon\left(Q_1^\varepsilon(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)\right) + \chi_\varepsilon\left(Q_2^\varepsilon(\chi_\varepsilon\gamma_\alpha^\varepsilon)\right) + Q_3^\varepsilon. \tag{44}$$

In some of the terms the mollification operator χ^ε appears twice; the reason for this is so that we can perform integration by parts in the energy estimate. The placement of mollifiers will become clear in the proof of Theorem 10. At this point, we have almost completely specified the mollified system. What remains now is to give the definition of some of the mollified versions of the auxiliary quantities, such as V_W^ε and Q_1^ε, among others.

$$V_W^\varepsilon = \partial_\alpha^{-1}\left(\frac{\pi}{L^\varepsilon}H((\chi_\varepsilon\gamma^\varepsilon)(\chi_\varepsilon\theta_\alpha^\varepsilon)) - \mathbb{P}(\mathbf{m}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)\right). \tag{45}$$

$$Q_1^\varepsilon = \frac{3}{2}(\chi_\varepsilon\theta_\alpha^\varepsilon)^2 - \frac{(L^\varepsilon)^2\bar{c}_1}{4\pi^2}, \tag{46}$$

We define $\mathbf{m}^\varepsilon$ the same way that $\mathbf{m}$ is defined in (23), but we use the mollified quantities instead:

$$\Phi(\mathbf{m}^\varepsilon)^* = z_\alpha^\varepsilon\mathcal{K}[z_d^\varepsilon]\left(\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right) + \frac{z_\alpha^\varepsilon}{2i}\left[H, \frac{1}{(z_\alpha^\varepsilon)^2}\right]\left(z_\alpha^\varepsilon\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right). \tag{47}$$

The mollified Birkhoff-Rott integral, $\mathbf{W}^\varepsilon$, is similarly defined the same way as $\mathbf{W}$, but in terms of the new quantities. We have

$$\Phi(\mathbf{W}^\varepsilon)^* = \frac{1}{4\pi i}\mathrm{PV}\int_0^{2\pi}\gamma^\varepsilon(\alpha')\cot\left(\frac{1}{2}(z_d^\varepsilon(\alpha) - z_d^\varepsilon(\alpha'))\right)\ d\alpha'.$$

Then, we define U^ε to be

$$U^\varepsilon = \mathbf{W}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon.$$

We also need the mollified version of (22)

$$\mathbf{W}_\alpha^\varepsilon = \frac{\pi}{L^\varepsilon}H(\gamma_\alpha^\varepsilon)\hat{\mathbf{n}}^\varepsilon - \frac{\pi}{L^\varepsilon}H(\gamma^\varepsilon\theta_\alpha^\varepsilon)\hat{\mathbf{t}}^\varepsilon + \mathbf{m}^\varepsilon, \tag{48}$$

We let V^ε be the mollified version of V, which was defined in (15):

$$V^\varepsilon = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha^\varepsilon U^\varepsilon) + V^\varepsilon(0,t). \tag{49}$$

We next define Q_2^ε and Q_3^ε. Of these, Q_2^ε is straightforward and Q_3^ε will take some effort. We let Q_2^ε be given by

$$Q_2^\varepsilon = \frac{2\pi V_W^\varepsilon}{L^\varepsilon} - \frac{2A\pi^2}{(L^\varepsilon)^2}\gamma^\varepsilon. \tag{50}$$

To define Q_3^ε, we first need to rewrite, again, the γ_t equation (to deal with the fact that it is actually an integral equation). We write it as

$$\gamma_t = -2A\mathcal{J}[z_d]\gamma_t + \Xi,$$

where

$$\Xi = -\widetilde{S}\partial_\alpha^4\theta - \widetilde{S}Q_1 + \gamma_\alpha Q_2 + \widetilde{Q}_3,$$

with $\widetilde{Q}_3$ given by

$$\widetilde{Q}_3 = \frac{2\pi\gamma\partial_\alpha V_W}{L} - 2AR_2 - 2AR_3 + 2AV_W\mathbf{m}\cdot\hat{\mathbf{t}} - 2Ay_\alpha.$$

Then, we can solve for γ_t :

$$\gamma_t = (I + 2A\mathcal{J}[z_d])^{-1}\Xi.$$

We can then rewrite Q_3 as

$$Q_3 = \widetilde{Q}_3 - 2A\mathcal{J}[z_d](I + 2A\mathcal{J}[z_d])^{-1}\Xi.$$

We can then define $\widetilde{Q}_3^\varepsilon$ and Ξ^ε :

$$\widetilde{Q}_3^\varepsilon = \frac{2\pi\gamma^\varepsilon V_W^\varepsilon}{L^\varepsilon} - 2AR_2^\varepsilon - 2AR_3^\varepsilon + 2AV_W^\varepsilon\mathbf{m}^\varepsilon\hat{\mathbf{t}}^\varepsilon - 2Ay_\alpha^\varepsilon,$$

$$\Xi^\varepsilon = -\widetilde{S}^\varepsilon\partial_\alpha^4\theta^\varepsilon - \widetilde{S}^\varepsilon Q_1^\varepsilon + \gamma_\alpha^\varepsilon Q_2^\varepsilon + \widetilde{Q}_3^\varepsilon.$$

This, naturally, still leaves us needing to define R_2^ε and R_3^ε; we will do this in a moment. First, however, we define Q_3^ε as

$$Q_3^\varepsilon = \widetilde{Q}_3^\varepsilon - 2A\mathcal{J}[z_d^\varepsilon](I + 2A\mathcal{J}[z_d^\varepsilon])^{-1}\Xi^\varepsilon. \tag{51}$$

For R_3, the original statement of its definition involved L_t and $z_{t\alpha}$; we now rewrite R_3 by substituting for these:

$$R_3 = \mathrm{Re}\left\{-\frac{\pi z_\alpha}{L}\left[H, \frac{V_W}{z_\alpha}\right](\gamma\theta_\alpha)\right\} + \mathrm{Re}\left\{-\frac{\pi}{z_\alpha}H\left(\frac{\gamma\mathbf{m}\cdot\hat{\mathbf{n}}}{z_\alpha}\right)\right\} + \mathrm{Re}\left\{\frac{z_\alpha\left(\int_0^{2\pi}\theta_\alpha U\ d\alpha\right)}{2Li}H\left(\frac{\gamma}{z_\alpha}\right)\right\} - \mathrm{Re}\left\{z_\alpha\mathcal{K}[z_d]\left(\frac{\gamma\partial_\alpha(U\hat{\mathbf{n}} + V\hat{\mathbf{t}})}{z_\alpha}\right)\right\}.$$

Defining R_3^ε is then straightforward:

$$R_3^\varepsilon = \mathrm{Re}\left\{-\frac{\pi z_\alpha^\varepsilon}{L^\varepsilon}\left[H, \frac{V_W^\varepsilon}{z_\alpha^\varepsilon}\right](\gamma^\varepsilon\theta_\alpha^\varepsilon)\right\} + \mathrm{Re}\left\{-\frac{\pi}{z_\alpha^\varepsilon}H\left(\frac{\gamma^\varepsilon \mathbf{m}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon}{z_\alpha^\varepsilon}\right)\right\}$$
$$+ \mathrm{Re}\left\{\frac{z_\alpha^\varepsilon\left(\int_0^{2\pi}\theta_\alpha^\varepsilon U^\varepsilon \, d\alpha\right)}{2L^\varepsilon i}H\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)\right\} - \mathrm{Re}\left\{z_\alpha^\varepsilon \mathcal{K}[z_d^\varepsilon]\left(\frac{\gamma^\varepsilon\partial_\alpha(U^\varepsilon\hat{\mathbf{n}}^\varepsilon + V^\varepsilon\hat{\mathbf{t}}^\varepsilon}{(z_\alpha^\varepsilon)}\right)\right\}. \quad (52)$$

For R_2, we rewrite (26) by substituting for z_t :

$$R_2 = -\mathrm{Re}\left\{\frac{z_\alpha}{2i}[H, U\hat{\mathbf{n}} + V\hat{\mathbf{t}}]\left(\frac{1}{z_\alpha}\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\} + \mathrm{Re}\left\{z_\alpha(U\hat{\mathbf{n}} + V\hat{\mathbf{t}})\mathcal{K}[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\}$$
$$- \mathrm{Re}\left\{z_\alpha\mathcal{K}[z_d]\left((U\hat{\mathbf{n}} + V\hat{\mathbf{t}})\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)\right\}.$$

Defining R_2^ε is then straightforward:

$$R_2^\varepsilon = -\mathrm{Re}\left\{\frac{z_\alpha^\varepsilon}{2i}[H, U^\varepsilon\hat{\mathbf{n}}^\varepsilon + V^\varepsilon\hat{\mathbf{t}}^\varepsilon]\left(\frac{1}{z_\alpha^\varepsilon}\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right)\right\} + \mathrm{Re}\left\{z_\alpha^\varepsilon(U^\varepsilon\hat{\mathbf{n}}^\varepsilon + V^\varepsilon\hat{\mathbf{t}}^\varepsilon)\mathcal{K}[z_d^\varepsilon]\left(\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right)\right\}$$
$$- \mathrm{Re}\left\{z_\alpha^\varepsilon\mathcal{K}[z_d^\varepsilon]\left((U^\varepsilon\hat{\mathbf{n}}^\varepsilon + V^\varepsilon\hat{\mathbf{t}}^\varepsilon)\left(\frac{\gamma^\varepsilon}{z_\alpha^\varepsilon}\right)_\alpha\right)\right\}. \quad (53)$$

5.2 Auxiliary estimates

In this section, we give estimates for quantities like $\mathbf{W}^\varepsilon$ or V_W^ε, in terms of norms of θ^ε and γ^ε. We note that the same estimates apply when there is no regularization, i.e., when $\varepsilon = 0$, and the proof is exactly the same. The estimates on regularized quantitites is used in the proof of Theorem 10, while those for unregularized quantities is used in the proof of Theorem 12.

Lemma 7 *Let* $(\theta^\varepsilon, \gamma^\varepsilon) \in \mathcal{O}$ *be given, such that* θ^ε *satisfies* $\langle\langle \sin(\theta^\varepsilon)\rangle\rangle = 0$. *Then, the following estimates are satisfied:*

$$\|z_\alpha^\varepsilon\|_{H^s} \le c(1 + \|\theta^\varepsilon\|_{H^s}), \quad (54)$$
$$\|z_d^\varepsilon\|_{H^{s+1}} \le c(1 + \|\theta^\varepsilon\|_{H^s}), \quad (55)$$
$$\|\hat{\mathbf{t}}^\varepsilon\|_{H^s} \le c(1 + \|\theta^\varepsilon\|_{H^s}) \quad (56)$$
$$\|\hat{\mathbf{n}}^\varepsilon\|_{H^s} \le c(1 + \|\theta^\varepsilon\|_{H^s}) \quad (57)$$
$$\|\mathbf{m}^\varepsilon\|_{H^s} \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}} \exp\{c_2\|\theta^\varepsilon\|_{H^s}\}, \quad (58)$$
$$\|U^\varepsilon\|_{H^{s-3/2}} \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}} \exp\{c_2\|\theta^\varepsilon\|_{H^s}\}, \quad (59)$$
$$\|\mathbf{W}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon\|_{H^{s-1/2}} \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}} \exp\{c_2\|\theta^\varepsilon\|_{H^s}\}, \quad (60)$$
$$\|V_W^\varepsilon\|_{H^{s-1/2}} \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}} \exp\{c_2\|\theta^\varepsilon\|_{H^s}\}, \quad (61)$$
$$\|V^\varepsilon\|_{H^{s-1/2}} \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}} \exp\{c_2\|\theta^\varepsilon\|_{H^s}\}, \quad (62)$$
$$\|Q_1^\varepsilon\|_{H^{s-1}} \le c(1 + \|\theta^\varepsilon\|_{H^s}), \quad (63)$$

$$\|Q_2^\varepsilon\|_{H^{s-3/2}} \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}}\exp\{c_2\|\theta^\varepsilon\|_{H^s}\}, \tag{64}$$

$$\|Q_3^\varepsilon\|_{H^{s-3/2}} \le c_1\left(\|\gamma^\varepsilon\|^2_{H^{s-3/2}}+1\right)\exp\{c_2\|\theta^\varepsilon\|_{H^s}\}, \tag{65}$$

$$|\mu^\varepsilon| \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}}\exp\{c_2\|\theta^\varepsilon\|_{H^s}\}. \tag{66}$$

where the constants are independent of ε.
Proof: The estimate (54) follows immediately from (37) and (38), together with (a) a standard composition estimate [37], and (b) the fact that the definition of $\mathcal{O}$ includes a bound on the length. The estimates (56) and (57), in light of (40), are similar. Since z_d^ε is defined in (39) by integrating x_α^ε and y_α^ε, the estimate (55) follows.

To establish (58), we use Lemma 2 to bound the first term on the right-hand side of (47), and we use Lemma 5 to bound the second term on the right-hand side of (47). To establish (58), we also rely on (54) and (55). The estimate (59) follows from (48) and the bound (58) on $\mathbf{m}^\varepsilon$. To establish the estimate on $\mathbf{W}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon$, we use $(\mathbf{W}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)_\alpha = \mathbf{W}^\varepsilon_\alpha\cdot\hat{\mathbf{t}}^\varepsilon + U^\varepsilon\theta^\varepsilon_\alpha$; the estimate then follows by taking the tangential component of (48) and using the previous estimate on U^ε. The estimates on V_W^ε and V^ε readily follow from (45) and (49) and the bound on $\mathbf{m}^\varepsilon$. The estimates of Q_1^ε and Q_2^ε are easily obtained from their definitions (46) and (50) using prior estimates.

The estimate on Q_3^ε defined in (51) is the most involved. We first obtain bounds on R_2^ε and R_3^ε of the form $\|R_i^\varepsilon\|_{H^{s-3/2}} \le c_1\|\gamma^\varepsilon\|_{H^{s-3/2}}\exp\{c_2\|\theta^\varepsilon\|_{H^s}\}$ for $i=2,\ 3$. The estimate on R_2^ε follows from using Lemma 5 to bound the first term on the right hand side of (53) and Lemma 2 to bound the second and third terms. The estimate on R_3^ε uses Lemma 4 to bound the first term on the right hand side of (52) and Lemma 2 to bound the fourth term; estimates on the second and third terms are straighforward, with the Schwartz inequality used to obtain a bound on the integral quantity $\int_0^{2\pi}\theta_\alpha U\ d\alpha$. Proceeding with the estimate on Q_3^ε, we use previous estimates to obtain a bound on the first term in (51):

$$\|\tilde{Q}_3^\varepsilon\|_{H^{s-3/2}} \le c_1\left(\|\gamma^\varepsilon\|^2_{H^{s-3/2}}\exp\{c_2\|\theta^\varepsilon\|_{H^s}\}+\|\theta^\varepsilon\|_{H^s}\right). \tag{67}$$

Next, since we have assumed s is sufficiently large so that $\Xi^\varepsilon\in H^0$, Lemma 6 implies that the second term in (51) satisfies $(I+2A\mathcal{J}[z_d^\varepsilon])^{-1}\Xi^\varepsilon = \gamma_t^\varepsilon \in H^0$, and in particular $\|\gamma_t^\varepsilon\|_0 \le c$. We then find an estimate on the second term in (51), $\mathcal{J}[z_d^\varepsilon]\gamma_t^\varepsilon$, from H_0 to the higher norm H_{s-1}. To obtain this estimate, we write

$$\begin{aligned}\mathcal{J}[z_d^\varepsilon]\gamma_t^\varepsilon &= \mathrm{Re}\left(\frac{iz_\alpha^\varepsilon}{4\pi}\mathrm{PV}\int_0^{2\pi}\gamma_t^{\varepsilon\prime}\cot\frac{z_d^\varepsilon-z_d^{\varepsilon\prime}}{2}\ d\alpha'\right)\\ &= \mathrm{Re}\left(-z_\alpha^\varepsilon\mathcal{K}[z_d^\varepsilon]\gamma_t^\varepsilon + iz_\alpha^\varepsilon[H,\frac{1}{z_\alpha^\varepsilon}](\gamma_t^\varepsilon)\right),\end{aligned}$$

and then apply Lemmas 2 and 4 to get

$$\begin{aligned}\|\mathcal{J}[z_d^\varepsilon]\gamma_t^\varepsilon\|_{H_{s-1}} &\le c\|\gamma_t^\varepsilon\|_{H_0}\exp\{c\|\theta^\varepsilon\|_{H_s}\}\\ &\le c_1\exp\{c\|\theta^\varepsilon\|_{H_s}\}.\end{aligned} \tag{68}$$

Combining (67) with (68) gives the estimate (65). Finally, the estimate on μ^ε is readily obtained from its definition (42) using previous estimates and the Schwartz inequality.

5.3 The energy estimate

We state the Picard theorem for ordinary differential equations on a Banach space; the particular statement we quote is from [31], and a similar statement can be found in [44].

Theorem 8 (Picard) *Let $O \subseteq B$ be an open subset of a Banach space, B. Let $F : O \to B$. Assume that F is locally Lipschitz continuous, i.e., that for all $x \in O$, there exists an open neighborhood of x, $U_x \subseteq O$, and $c > 0$ such that for all $x_1, x_2 \in U_x$,*

$$\|F(x_1) - F(x_2)\|_B \le c\|x_1 - x_2\|_B.$$

Then, for every $x_0 \in O$, there exists $T > 0$ and $x \in C^1((-T,T);O)$ such that x is the solution of the initial value problem

$$\frac{dx}{dt} = F(x), \qquad x(0) = x_0.$$

In order to use the Picard theorem, we introduce the following open set, $\mathcal{O}$. Let positive constants $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$ be given. We let $\mathcal{O}$ be the subset of $H^s \times H^{s-3/2}$ such that for all $(f_1, f_2) \in \mathcal{O}$, the following three conditions are satisfied:

$$\|(f_1, f_2)\|_{H^s \times H^{s-3/2}} < \bar{d}_1, \qquad L[f_1] < \bar{d}_2, \tag{69}$$

$$|q_1[f_1](\alpha, \alpha')| > \bar{d}_3, \quad \forall \alpha, \alpha'. \tag{70}$$

The conditions in (69) state that the set $\mathcal{O}$ is a subset of an open ball, such that the curves generated by tangent angle f_1 all have bounded length. The condition in (70) states that the curves generated by tangent angle f_1 are not close to self-intersection.

Theorem 9 *Let $(\theta_0, \gamma_0) \in \mathcal{O}$ be given, with θ_0 satisfying $\langle\langle \sin(\theta_0) \rangle\rangle = 0$. There exists $T_\varepsilon > 0$ and $(\theta^\varepsilon, \gamma^\varepsilon) \in C^1((-T^\varepsilon, T^\varepsilon); \mathcal{O})$ such that $(\theta^\varepsilon, \gamma^\varepsilon)$ is the unique solution of the initial value problem given by (41) with initial data (θ_0, γ_0).*

We have thus demonstrated the existence of solutions to the mollified system. We would like to pass to the limit as $\varepsilon \to 0^+$. However, we cannot do this yet, as the time interval from Theorem 9 could go to zero as ε vanishes. Our next step is to prove an energy estimate, uniformly in ε, for the solutions $(\theta^\varepsilon, \gamma^\varepsilon)$. We can then use the continuation theorem for ordinary differential equations on a Banach space to find that the solutions of the mollified system exist on a common time interval; after this, we will be able to pass to the limit as the regularization vanishes.

Theorem 10 *Let $(\theta_0, \gamma_0) \in \mathcal{O}$ be given, with θ_0 satisfying $\langle\langle \sin(\theta_0) \rangle\rangle = 0$. Let $\varepsilon > 0$ be given. Let $(\theta^\varepsilon, \gamma^\varepsilon) \in C([0,T]; \mathcal{O})$ be a solution of (41), with initial conditions (θ_0, γ_0). (Note that this T may depend on ε.) Then there exist constants $c_1 \in (0, \infty)$, $c_2 \in (0,1)$, and $c_3 \in (0, \infty)$, depending only on s, $\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$, $\|\theta_0\|_{H^s}$, and $\|\gamma_0\|_{H^{s-3/2}}$, such that*

$$\|\theta^\varepsilon\|^2_{H^s} + \|\gamma^\varepsilon\|^2_{H^{s-3/2}} \le -c_1 \ln(c_2 - c_3 t). \tag{71}$$

Proof: We will define an energy functional, E, such that

$$\frac{1}{2}\left(\|\theta^\varepsilon\|^2_s + \|\gamma^\varepsilon\|^2_{s-3/2}\right) \le E, \tag{72}$$

and such that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{dE}{dt} \leq C_1 \exp\{C_2 E\}. \tag{73}$$

The inequality (73) implies

$$E(t) \leq -\frac{-\ln(e^{-C_2 E(0)} - C_1 C_2 t)}{C_2}. \tag{74}$$

In light of (72), and renaming the constants, we see that (71) is then satisfied. Thus, we need only to establish (73). Throughout the proof, we make frequent use of the properties of the mollifier of Ξ_ε, i.e., that it is self-adjoint and commutes with derivatives.

As in the example of Section 2, we will define the energy a bit at a time. The energy will be given by

$$E = E_0 + E_1 + E_2 + E_3 + E_4 + E_5,$$

where we will now give the definition of E_0, E_1, and E_2. The rest of the terms will be defined as needed. We have

$$E_0 = \frac{1}{2}\int_0^{2\pi} (\theta^\varepsilon)^2 + (\gamma^\varepsilon)^2 \ d\alpha,$$

$$E_1 = \frac{c_1(t)}{2}\int_0^{2\pi} (\partial_\alpha^s \theta^\varepsilon)^2 \ d\alpha,$$

$$E_2 = \frac{c_2(t)}{2}\int_0^{2\pi} (\partial_\alpha^{s-2}\gamma^\varepsilon)\Lambda(\partial_\alpha^{s-2}\gamma^\varepsilon) \ d\alpha.$$

We see that property (72) is satisfied as long as $c_1(t) \geq 1$, $c_2(t) \geq 1$, for all t, and $E_i \geq 0$ for $i \in \{3,4,5\}$. When we choose c_1, c_2, E_3, E_4, and E_5, all of these properties will be satisfied.

To begin, we take the time derivative of E_0 :

$$\frac{dE_0}{dt} = \int_0^{2\pi} \theta^\varepsilon \theta_t^\varepsilon + \gamma^\varepsilon \gamma_t^\varepsilon \ d\alpha.$$

Since s is sufficiently large, it is immediate, from the evolution equation (41), the definitions (42), (43), (44), and related equations, as well as the estimates of Section 5.2, that this can be bounded in terms of the energy:

$$\frac{dE_0}{dt} \leq C_1 \exp\{C_2 E\}.$$

We next take the time derivative of E_1 :

$$\frac{dE_1}{dt} = \frac{dc_1}{dt}\cdot\frac{1}{2}\int_0^{2\pi} (\partial_\alpha^s\theta^\varepsilon)^2 \ d\alpha + c_1\int_0^{2\pi} (\partial_\alpha^s\theta^\varepsilon)(\partial_\alpha^s\theta_t^\varepsilon) \ d\alpha. \tag{75}$$

We note that $\partial_\alpha \mu^\varepsilon = 0$, so there is no contribution from μ^ε in (75). To proceed with (75), we will write a formula for $\partial_\alpha^s \theta_t^\varepsilon$. Applying ∂_α^s to (43), we get

$$\partial_\alpha^s \theta_t^\varepsilon = \frac{2\pi^2}{(L^\varepsilon)^2}\chi_\varepsilon H(\partial_\alpha^{s+1}\gamma^\varepsilon) + \frac{2\pi}{L^\varepsilon}\left(\chi_\varepsilon(\partial_\alpha^s V_W^\varepsilon)(\chi_\varepsilon \theta_\alpha^\varepsilon)\right) + \frac{2\pi}{L^\varepsilon}\chi_\varepsilon\left(V_W^\varepsilon(\chi_\varepsilon \partial_\alpha^{s+1}\theta^\varepsilon)\right) + \Phi_1, \tag{76}$$

where Φ_1 is given by the formula

$$\Phi_1 = \frac{2\pi}{L^\varepsilon}\partial_\alpha^s(\mathbf{m}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon) + \frac{2\pi}{L^\varepsilon}\chi_\varepsilon\left(\sum_{j=1}^{s-1}\binom{s}{j}(\partial_\alpha^j V_W^\varepsilon)(\chi_\varepsilon\partial_\alpha^{s-j+1}\theta^\varepsilon)\right).$$

All of the summands here involve at most $s-1$ derivatives of V_W^ε, and at most s derivatives of θ^ε. Therefore, the estimates of Section 5.2 and (72) immediately imply that $\|\Phi_1\|_0 \le C_1\exp\{C_2 E\}$.

We then plug (76) into (75). Using the property that the mollifier χ_ε is self-adjoint and commutes with derivatives, we compute the following:

$$\frac{dE_1}{dt} = \frac{2\pi^2 c_1}{(L^\varepsilon)^2}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\chi_\varepsilon H\partial_\alpha^{s+1}\gamma^\varepsilon)\ d\alpha + \frac{2\pi c_1}{L^\varepsilon}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\partial_\alpha^s V_W^\varepsilon)(\chi_\varepsilon\theta_\alpha^\varepsilon)\ d\alpha$$
$$+\frac{2\pi c_1}{L^\varepsilon}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\chi_\varepsilon\partial_\alpha^{s+1}\theta^\varepsilon)V_W^\varepsilon\ d\alpha + \frac{dc_1}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\partial_\alpha^s\theta^\varepsilon)^2\ d\alpha + \int_0^{2\pi}(\partial_\alpha^s\theta^\varepsilon)\Phi_1\ d\alpha. \quad (77)$$

For the third term on the right-hand side of (77), we recognize a perfect derivative and integrate by parts:

$$\frac{dE_1}{dt} = \frac{2\pi^2 c_1}{(L^\varepsilon)^2}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\chi_\varepsilon H\partial_\alpha^{s+1}\gamma^\varepsilon)\ d\alpha + \frac{2\pi c_1}{L^\varepsilon}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\partial_\alpha^s V_W^\varepsilon)(\chi_\varepsilon\theta_\alpha^\varepsilon)\ d\alpha + \Psi_1, \quad (78)$$

where Ψ_1 is defined as

$$\Psi_1 = -\frac{\pi c_1}{L^\varepsilon}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)^2\partial_\alpha V_W^\varepsilon\ d\alpha + \frac{dc_1}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\partial_\alpha^s\theta^\varepsilon)^2\ d\alpha + c_1\int_0^{2\pi}(\partial_\alpha^s\theta^\varepsilon)\Phi_1\ d\alpha. \quad (79)$$

Before we are able to choose c_1, we must compute $\frac{dE_2}{dt}$, which we now do. To begin, we have simply

$$\frac{dE_2}{dt} = \frac{dc_2}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\partial_\alpha^{s-2}\gamma^\varepsilon)\Lambda(\partial_\alpha^{s-2}\gamma^\varepsilon)\ d\alpha + c_2\int_0^{2\pi}(\partial_\alpha^{s-2}\gamma_t^\varepsilon)(H\partial_\alpha^{s-1}\gamma^\varepsilon)\ d\alpha. \quad (80)$$

We next must compute $\partial_\alpha^{s-2}\gamma_t^\varepsilon$:

$$\partial_\alpha^{s-2}\gamma_t^\varepsilon = -\widetilde{S}^\varepsilon\chi_\varepsilon\partial_\alpha^{s+2}\theta^\varepsilon - \widetilde{S}^\varepsilon\chi_\varepsilon\left((\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)Q_1^\varepsilon\right) + \chi_\varepsilon\left((\chi_\varepsilon\partial_\alpha^{s-1}\gamma^\varepsilon)Q_2^\varepsilon\right) + \Phi_2, \quad (81)$$

where Φ_2 is defined as

$$\Phi_2 = -\sum_{j=0}^{s-3}\binom{s-2}{j}\widetilde{S}^\varepsilon\chi_\varepsilon\left((\partial_\alpha^{j+2}\theta^\varepsilon)(\partial_\alpha^{s-2-j}Q_1^\varepsilon)\right) + \sum_{j=0}^{s-3}\binom{s-2}{j}\chi_\varepsilon\left((\chi_\varepsilon\partial_\alpha^{j+1}\gamma^\varepsilon)(\partial_\alpha^{s-2-j}Q_2^\varepsilon)\right) + \partial_\alpha^{s-2}Q_3^\varepsilon.$$

From the estimates of Section 5.2, since the summands here involve at most $s-1$ derivatives of θ^ε, at most $s-2$ derivatives of γ^ε, and at most $s-2$ derivatives of Q_i for $i\in\{1,2,3\}$, the estimates of Section 5.2 imply $\|\Phi_2\|_{1/2} \le C_1\exp\{C_2 E\}$.

We plug (81) into (80), and we again repeatedly use the fact that χ_ε is self-adjoint (as well as the fact that $\chi_\varepsilon = \chi_\varepsilon^2$) :

$$\frac{dE_2}{dt} = -c_2\widetilde{S}^\varepsilon\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^{s+2}\theta^\varepsilon)\ d\alpha - c_2\widetilde{S}^\varepsilon\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)Q_1^\varepsilon\ d\alpha$$
$$+c_2\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^{s-1}\gamma^\varepsilon)Q_2^\varepsilon\ d\alpha + \frac{dc_2}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\partial_\alpha^{s-2}\gamma^\varepsilon)\Lambda(\partial_\alpha^{s-2}\gamma^\varepsilon)\ d\alpha + c_2\int_0^{2\pi}(\Lambda\partial_\alpha^{s-2}\gamma^\varepsilon)\Phi_2\ d\alpha. \quad (82)$$

We integrate the first term on the right-hand side of (82) by parts twice, and we recall the definition of $\widetilde{S}^\varepsilon$:

$$\frac{dE_2}{dt} = -\frac{8\pi^3 S c_2}{(L^\varepsilon)^3}\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s+1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)\ d\alpha + c_2\widetilde{S}^\varepsilon\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)Q_1^\varepsilon\ d\alpha + \Psi_2, \quad (83)$$

where Ψ_2 is defined as

$$\Psi_2 = c_2\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^{s-1}\gamma^\varepsilon)Q_2^\varepsilon\ d\alpha + \frac{dc_2}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\partial_\alpha^{s-2}\gamma^\varepsilon)\Lambda(\partial_\alpha^{s-2}\gamma^\varepsilon)\ d\alpha \\ + c_2\int_0^{2\pi}(\Lambda\partial_\alpha^{s-2}\gamma^\varepsilon)\Phi_2\ d\alpha. \quad (84)$$

We now add (78) and (83), and group the like terms:

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} = \left[\frac{2\pi^2 c_1}{(L^\varepsilon)^2} - \frac{8\pi^3 S c_2}{(L^\varepsilon)^3}\right]\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s+1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)\ d\alpha \\ + \frac{2\pi c_1}{L^\varepsilon}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\partial_\alpha^s V_W^\varepsilon)(\chi_\varepsilon\theta_\alpha^\varepsilon)\ d\alpha - c_2\widetilde{S}^\varepsilon\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)Q_1^\varepsilon\ d\alpha + \Psi_1 + \Psi_2. \quad (85)$$

To make the first term on the right-hand side of (85) vanish, we choose c_1 and c_2 to satisfy

$$\frac{2\pi^2 c_1}{(L^\varepsilon)^2} - \frac{8\pi^3 S c_2}{(L^\varepsilon)^3} = 0;$$

this is satisfied as long as

$$\frac{c_1}{c_2} = \frac{4\pi S}{L^\varepsilon}.$$

Recall also the previous conditions $c_1 \geq 1$ and $c_2 \geq 1$. Therefore, we choose

$$c_2 = \max\left\{L^\varepsilon, \frac{L^\varepsilon}{S}\right\}, \qquad c_1 = \frac{4\pi S c_2}{L^\varepsilon}. \quad (86)$$

Clearly $c_2 \geq 1$ by definition; for c_1, it follows that $c_1(t) \geq 1$ for all t since the horizontal periodicity of the interface implies $L^\varepsilon(t) \geq 2\pi$ for all t. Having made these choices for c_1 and c_2, we have

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} = \frac{2\pi c_1}{L^\varepsilon}\int_0^{2\pi}(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\partial_\alpha^s V_W^\varepsilon)(\chi_\varepsilon\theta_\alpha^\varepsilon)\ d\alpha - c_2\widetilde{S}^\varepsilon\int_0^{2\pi}(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)Q_1^\varepsilon\ d\alpha + \Psi_1 + \Psi_2. \quad (87)$$

The integrals on the right-hand side of (87) cannot be estimated in terms of the energy; we must choose E_3 to achieve an additional cancellation. First, however, we will rewrite (87) in order to more fully understand the terms which cannot be bounded in terms of the energy.

Applying ∂_α^s to V_W^ε, using (45), and extracting the leading-order term, we have

$$\partial_\alpha^s V_W^\varepsilon = \frac{\pi}{L^\varepsilon}(\chi_\varepsilon\theta_\alpha^\varepsilon)H(\chi_\varepsilon\partial_\alpha^{s-1}\gamma^\varepsilon) + \frac{\pi}{L^\varepsilon}\left[H, \chi_\varepsilon\theta_\alpha^\varepsilon\right](\chi_\varepsilon\partial_\alpha^{s-1}\gamma^\varepsilon) \\ + \frac{\pi}{L^\varepsilon}\sum_{j=0}^{s-2}\binom{s-1}{j}H\left((\chi_\varepsilon\partial_\alpha^j\gamma^\varepsilon)(\chi_\varepsilon\partial_\alpha^{s-j}\theta^\varepsilon)\right) - \partial_\alpha^s\left(\mathbf{m}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon\right). \quad (88)$$

We substitute (88) and (46) into (87), and we also use (86) to substitute for c_1. We collect like terms, and we collect lower-order terms, arriving at the following:

$$\frac{dE_1}{dt} + \frac{dE_2}{dt} = c_2 \int_0^{2\pi} \left[\frac{2\pi S\bar{c}_1}{L^\varepsilon} - \frac{4\pi^3 S}{(L^\varepsilon)^3}(\chi_\varepsilon \theta_\alpha^\varepsilon)^2 \right] (\chi_\varepsilon \partial_\alpha^s \theta^\varepsilon)(\chi_\varepsilon H \partial_\alpha^{s-1}\gamma^\varepsilon)\ d\alpha + \Psi_3, \tag{89}$$

where Ψ_3 is given by

$$\begin{aligned}\Psi_3 = \Psi_1 + \Psi_2 + \frac{8\pi^3 Sc_2}{(L^\varepsilon)^3} \int_0^{2\pi} (\chi_\varepsilon \partial_\alpha^s \theta^\varepsilon)(\chi_\varepsilon \theta_\alpha^\varepsilon) \left(\sum_{j=0}^{s-2} \binom{s-1}{j} H\left((\chi_\varepsilon \partial_\alpha^j \gamma^\varepsilon)(\chi_\varepsilon \partial_\alpha^{s-j}\theta^\varepsilon) \right) \right) d\alpha \\ + \frac{8\pi^3 Sc_2}{(L^\varepsilon)^3} \int_0^{2\pi} (\chi_\varepsilon \partial_\alpha^s \theta^\varepsilon)(\chi_\varepsilon \theta_\alpha^\varepsilon) \left([H, \chi_\varepsilon \theta_\alpha^\varepsilon] (\chi_\varepsilon \partial_\alpha^{s-1}\gamma^\varepsilon) \right) d\alpha \\ - \frac{8\pi Sc_2}{(L^\varepsilon)^2} \int_0^{2\pi} (\chi_\varepsilon \partial_\alpha^s \theta^\varepsilon)(\chi_\varepsilon \theta_\alpha^\varepsilon)(\partial_\alpha^s (\mathbf{m}^\varepsilon \cdot \hat{\mathbf{t}}^\varepsilon))\ d\alpha.\end{aligned} \tag{90}$$

We define E_3 and E_4 as

$$E_3 = \frac{c_3(t)}{2} \int_0^{2\pi} (\chi_\varepsilon \partial_\alpha^{s-1} \theta^\varepsilon)^2\ d\alpha,$$

$$E_4 = \frac{1}{2} \int_0^{2\pi} \sqrt{c_4(\alpha,t)} (\partial_\alpha^{s-3}\gamma^\varepsilon) \Lambda \left(\sqrt{c_4(\alpha,t)} (\partial_\alpha^{s-3}\gamma^\varepsilon) \right) d\alpha.$$

Note that E_4 is in the form $\int g\Lambda g\ d\alpha$, so we may make use of (1) when taking its time derivative. We will specify $c_3(t)$ and $c_4(\alpha,t)$ in short order; these will satisfy $c_3 \geq 0$ and $c_4 \geq 1$, so that $E_3 \geq 0$, $E_4 \geq 0$, and also so that derivatives of $\sqrt{c_4}$ remain bounded. These conditions ensure that (72) is satisfied. In order to take the time derivatives of E_3 and E_4, it is again helpful to have formulas for spatial derivatives of θ_t^ε and γ_t^ε :

$$\partial_\alpha^{s-1}\theta_t^\varepsilon = \frac{2\pi^2}{(L^\varepsilon)^2} \chi_\varepsilon H \partial_\alpha^s \gamma^\varepsilon + \Phi_4, \tag{91}$$

$$\partial_\alpha^{s-3}\gamma_t^\varepsilon = -\widetilde{S}^\varepsilon \chi_\varepsilon \partial_\alpha^{s+1}\theta^\varepsilon + \Phi_5, \tag{92}$$

where Φ_4 and Φ_5 are given by

$$\Phi_4 = \frac{2\pi}{L^\varepsilon} \chi_\varepsilon \partial_\alpha^{s-1} \left(V_W^\varepsilon (\chi_\varepsilon \theta_\alpha^\varepsilon) \right) + \frac{2\pi}{L^\varepsilon} \partial_\alpha^{s-1} \left(\mathbf{m}^\varepsilon \cdot \hat{\mathbf{n}}^\varepsilon \right), \tag{93}$$

$$\Phi_5 = -\widetilde{S}^\varepsilon \chi_\varepsilon \partial_\alpha^{s-3} \left(Q_1^\varepsilon (\chi_\varepsilon \theta_{\alpha\alpha}^\varepsilon) \right) + \chi_\varepsilon \partial_\alpha^{s-3} \left(Q_2^\varepsilon (\chi_\varepsilon \gamma_\alpha^\varepsilon) \right) + \partial_\alpha^{s-3} Q_3^\varepsilon. \tag{94}$$

We compute the time derivative of E_3 :

$$\frac{dE_3}{dt} = \frac{dc_3}{dt} \cdot \frac{1}{2} \int_0^{2\pi} (\chi_\varepsilon \partial_\alpha^{s-1}\theta^\varepsilon)^2\ d\alpha + \int_0^{2\pi} c_3 (\chi_\varepsilon \partial_\alpha^{s-1}\theta^\varepsilon)(\chi_\varepsilon \partial_\alpha^{s-1}\theta_t^\varepsilon)\ d\alpha. \tag{95}$$

We substitute (91) into (95):

$$\frac{dE_3}{dt} = \frac{2\pi^2}{(L^\varepsilon)^2} \int_0^{2\pi} c_3 (\chi_\varepsilon \partial_\alpha^{s-1}\theta^\varepsilon)(\chi_\varepsilon H \partial_\alpha^s \gamma^\varepsilon)\ d\alpha + \Psi_4, \tag{96}$$

where Ψ_4 is defined as

$$\Psi_4 = \int_0^{2\pi} c_3(\chi_\varepsilon \partial_\alpha^{s-1}\theta^\varepsilon)(\chi_\varepsilon \Phi_4)\ d\alpha + \frac{dc_3}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\chi_\varepsilon \partial_\alpha^{s-1}\theta^\varepsilon)^2\ d\alpha. \tag{97}$$

Next, we compute the time derivative of E_4 :

$$\frac{dE_4}{dt} = \int_0^{2\pi} \frac{\partial c_4}{\partial t}\cdot\frac{1}{2\sqrt{c_4}}(\partial_\alpha^{s-3}\gamma^\varepsilon)\Lambda(\sqrt{c_4}(\partial_\alpha^{s-3}\gamma^\varepsilon))\ d\alpha + \int_0^{2\pi}\sqrt{c_4}(\partial_\alpha^{s-3}\gamma_t^\varepsilon)\Lambda(\sqrt{c_4}(\partial_\alpha^{s-3}\gamma^\varepsilon))\ d\alpha.$$

We comment now on the factor $\Lambda(\sqrt{c_4}(\partial_\alpha^{s-3}\gamma^\varepsilon))$. We rewrite this first by using the definition $\Lambda = H\partial_\alpha$, and the product rule:

$$\Lambda(\sqrt{c_4}(\partial_\alpha^{s-3}\gamma^\varepsilon)) = H(\sqrt{c_4}(\partial_\alpha^{s-2}\gamma^\varepsilon)) + H\left(\frac{1}{2\sqrt{c_4}}(\partial_\alpha c_4)(\partial_\alpha^{s-3}\gamma^\varepsilon)\right).$$

Next, for the first term on the right-hand side, we pull $\sqrt{c_4}$ through the Hilbert transform, incurring a commutator:

$$\Lambda(\sqrt{c_4}(\partial_\alpha^{s-3}\gamma^\varepsilon)) = \sqrt{c_4}H\partial_\alpha^{s-2}\gamma^\varepsilon + [H,\sqrt{c_4}](\partial_\alpha^{s-2}\gamma^\varepsilon) + H\left(\frac{1}{\sqrt{c_4}}(\partial_\alpha c_4)(\partial_\alpha^{s-3}\gamma^\varepsilon)\right).$$

Now, making use of this formula, and substituting from (92), we find the following:

$$\frac{dE_4}{dt} = -\frac{8c_4\pi^3 S}{(L^\varepsilon)^3}\int_0^{2\pi}(\chi_\varepsilon \partial_\alpha^{s+1}\theta^\varepsilon)(\chi^\varepsilon H\partial_\alpha^{s-2}\gamma^\varepsilon)\ d\alpha + \Psi_5, \tag{98}$$

where Ψ_5 is defined as

$$\begin{aligned}\Psi_5 = &\int_0^{2\pi}(\sqrt{c_4}\widetilde{S}^\varepsilon\chi_\varepsilon\partial_\alpha^{s+1}\theta^\varepsilon)H\left(\frac{1}{2\sqrt{c_4}}(\partial_\alpha c_4)(\partial_\alpha^{s-3}\gamma^\varepsilon)\right)\ d\alpha\\ &+\int_0^{2\pi}(\sqrt{c_4}\widetilde{S}^\varepsilon\chi_\varepsilon\partial_\alpha^{s+1}\theta^\varepsilon)[H,\sqrt{c_4}](\partial_\alpha^{s-2}\gamma^\varepsilon)\ d\alpha + \int_0^{2\pi}(\sqrt{c_4}\Phi_5)\Lambda(\sqrt{c_4}(\partial_\alpha^{s-3}\gamma^\varepsilon))\ d\alpha\\ &+\int_0^{2\pi}\frac{\partial c_4}{\partial t}\cdot\frac{1}{2\sqrt{c_4}}(\partial_\alpha^{s-3}\gamma^\varepsilon)\Lambda(\sqrt{c_4}(\partial_\alpha^{s-3}\gamma^\varepsilon))\ d\alpha.\end{aligned} \tag{99}$$

Next, we combine the time derivatives of E_1, E_2, E_3, and E_4. We integrate by parts in the integrals on the right-hand sides of (96) and (98), adding the results to (89):

$$\begin{aligned}&\frac{dE_1}{dt}+\frac{dE_2}{dt}+\frac{dE_3}{dt}+\frac{dE_4}{dt}\\ &= \int_0^{2\pi}\left[\frac{2\pi Sc_2\bar{c}_1}{L^\varepsilon} - \frac{4\pi^3 Sc_2}{(L^\varepsilon)^3}(\chi_\varepsilon\theta_\alpha^\varepsilon)^2 - \frac{2\pi^2 c_3}{(L^\varepsilon)^2} + \frac{8\pi^3 Sc_4}{(L^\varepsilon)^3}\right](\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\chi_\varepsilon H\partial_\alpha^{s-1}\gamma^\varepsilon)\\ &\quad + \frac{8\pi^3}{(L^\varepsilon)^3}\int_0^{2\pi}(\partial_\alpha c_4)(\chi_\varepsilon\partial_\alpha^s\theta^\varepsilon)(\chi_\varepsilon H\partial_\alpha^{s-2}\gamma^\varepsilon)\ d\alpha + \Psi_3+\Psi_4+\Psi_5.\end{aligned} \tag{100}$$

Then, we choose c_3 and c_4 so that the first integral on the right-hand side of (100) vanishes; recall that we have said $c_4 = c_4(\alpha,t)$; this is because our choice of c_4 will involve $(\chi_\varepsilon\theta_\alpha^\varepsilon)^2$. Recall further that we have stated that we will choose c_4 so that $c_4 \geq 1$; by bounding c_4 away from zero, we ensure that the factors of $\sqrt{c_4}$ which appear in denominators are not troublesome. Also recall that the constant $\bar{c}_1$ may be either positive or negative, while c_2 is positive. We write $\bar{c}_1$ using its positive and negative parts:

$$\bar{c}_1 = (\bar{c}_1)^+ - (\bar{c}_1)^-,$$

with these defined the usual way ($(\bar{c}_1)^+ = \max\{0, \bar{c}_1\}$ and $(\bar{c}_1)^- = -\min\{0, \bar{c}_1\}$, so that $(\bar{c}_1)^+ \geq 0$ and $(\bar{c}_1)^- \geq 0$). Then, we choose c_3 and c_4 so that $c_3 \geq 0$, $c_4 \geq 1$, and

$$\frac{2\pi S c_2 (\bar{c}_1)^+}{L^\varepsilon} - \frac{2\pi S c_2 (\bar{c}_1)^-}{L^\varepsilon} - \frac{4\pi^3 S c_2}{(L^\varepsilon)^3}(\chi_\varepsilon \theta^\varepsilon_\alpha)^2 - \frac{2\pi^2 c_3}{(L^\varepsilon)^2} + \frac{8\pi^3 S c_4}{(L^\varepsilon)^3} = 0.$$

We can accomplish our desired goals with the following choices of c_3 and c_4 :

$$c_3 = \frac{L^\varepsilon S c_2 (\bar{c}_1)^+}{\pi} + \frac{4\pi S}{L^\varepsilon},$$

$$c_4 = \frac{(L^\varepsilon)^2 c_2 (\bar{c}_1)^-}{4\pi^2} + \frac{c_2}{2}(\chi_\varepsilon \theta^\varepsilon_\alpha)^2 + 1.$$

The remaining terms on the right-hand side of (100) can then be bounded in terms of the energy. ■

5.4 Existence and regularity

We now state and complete the proof of our existence theorem.

Theorem 11 *Let $(\theta_0, \gamma_0) \in \mathcal{O}$ be given, with θ_0 satisfying $\langle\langle \sin(\theta_0) \rangle\rangle = 0$. There exists $T > 0$ and there exists $(\theta, \gamma) \in C([0,T]; \bar{\mathcal{O}})$ such that (θ, γ) satisfies (25), (20) with $(\theta(\cdot,0), \gamma(\cdot,0)) = (\theta_0, \gamma_0)$.*

Note that the set $\bar{\mathcal{O}}$ denotes the closure of the open set $\mathcal{O}$.

Proof: Theorem 10 implies that the norm of solutions of the mollified problem, $(\theta^\varepsilon, \gamma^\varepsilon)$, cannot immediately blow up; indeed, the estimate (71) indicates that the solutions $(\theta^\varepsilon, \gamma^\varepsilon)$ are bounded independently of ε. By the continuation theorem for autonomous ODEs on a Banach space [31], this implies that the solutions all exist on a common time interval. We conclude there exists $T > 0$ such that for all $\varepsilon > 0$, solutions of the initial value problem $(\theta^\varepsilon, \gamma^\varepsilon)$ are in $C([0,T]\mathcal{O})$. (We note that to draw this conclusion, we must also check that the time derivative of the length is bounded and that the chord-arc condition continues to be satisfied. We note that these conclusions also follow from (71), since the time derivatives of these quantities can be controlled by the norm of (θ, γ).)

We have proved that there exists $T > 0$ and $(\theta^\varepsilon, \gamma^\varepsilon) \in C([0,T]; \mathcal{O})$ which solve the mollified evolution equations (41), with this T independent of ε. Since $\mathcal{O}$ is a bounded subset of $H^s \times H^{s-3/2}$, and since we have taken s to be sufficiently large, this implies that each of $\theta^\varepsilon_\alpha$, θ^ε_t, $\gamma^\varepsilon_\alpha$, and γ^ε_t are uniformly bounded periodic functions. Thus, θ^ε and γ^ε are bounded, equicontinuous families. By the Arzela-Ascoli theorem, there exists $(\theta, \gamma) \in C([0,2\pi] \times [0,T]) \times C([0,2\pi] \times [0,T])$ such that a subsequence of $(\theta^\varepsilon, \gamma^\varepsilon)$ converges uniformly to (θ, γ) on $[0,2\pi] \times [0,T]$. We will now show that this pair, (θ, γ), is in the closure of $\mathcal{O}$, and also that it solves the non-mollified evolution equation, (32).

Since each of θ and γ are in $C([0,2\pi] \times [0,T])$, we see that they are also in $L^2([0,2\pi])$ at each time. By Lemma 1, we can conclude that the subsequence of $(\theta^\varepsilon, \gamma^\varepsilon)$ actually converges to (θ, γ) in $H^{s'} \times H^{s'-3/2}$, for any s' satisfying $3/2 \leq s' < s$. For any such s', this implies $(\theta, \gamma) \in L^\infty([0,T]; H^{s'} \times H^{s'-3/2})$.

Since the solutions θ^ε all satisfy (70), we can pass to the limit, finding

$$|q_1[\theta](\alpha,\alpha')| \geq \bar{d}_3 > 0, \quad \forall \alpha, \alpha'.$$

Furthermore, the solutions θ^ε all satisfy

$$\int_0^{2\pi} \sin(\theta^\varepsilon)\ d\alpha = 0;$$

since, along our subsequence, θ^ε converges uniformly to θ, we can pass to the limit, finding $\mathbb{P}(\sin(\theta)) = \sin(\theta)$. We can pass to the limit as ε vanishes (along our subsequence) in μ^ε as well. Recalling (42), and again in light of the regularity which we have already established, we see that

$$\lim_{\varepsilon\to 0^+} \mu^\varepsilon = -\frac{L}{4\pi^2}\int_0^{2\pi} \mathcal{B}\cos(\theta)\ d\alpha = -\frac{1}{2\pi}\int_0^{2\pi} (U_\alpha + V\theta_\alpha)\cos(\theta)\ d\alpha.$$

We have previously calculated this integral; it is equal to

$$\frac{L_t}{2\pi L}\int_0^{2\pi} \sin(\theta)\ d\alpha.$$

Since we have seen that this integral is equal to zero, we see that μ^ε vanishes as ε vanishes.

We now integrate (41) by integrating in time:

$$(\theta^\varepsilon, \gamma^\varepsilon) = (\theta_0, \gamma_0) + \int_0^t (\mathcal{B}_1^\varepsilon + \mu^\varepsilon, \mathcal{B}_2^\varepsilon)\ ds,$$

for any $t \in [0,T]$. We have established sufficient regularity thus far to be able to pass to the limit (along our subsequence) here, finding

$$(\theta, \gamma) = (\theta_0, \gamma_0) + \int_0^t (\mathcal{B}_1, \mathcal{B}_2)\ ds.$$

Differentiating this with respect to time, we have shown that (θ,γ) is indeed a solution of (32), as desired.

It remains to demonstrate that $(\theta,\gamma) \in C([0,T]; H^s \times H^{s-3/2})$. For any $t \in [0,T]$, the sequence $(\theta^\varepsilon(\cdot,t), \gamma^\varepsilon(\cdot,t))$ is uniformly bounded (with respect to both ε and t) in $H^s \times H^{s-3/2}$. Since the unit ball of a Hilbert space is weakly compact, there exists a weak limit along a (further) subsequence. However, this weak limit must clearly be (θ,γ); this implies that $(\theta,\gamma) \in L^\infty([0,T]; H^s \times H^{s-3/2})$.

To show continuity in time in $H^s \times H^{s-3/2}$, we must show that for any $t_* \in [0,T]$,

$$\lim_{t\to t_*} \|\theta(\cdot,t) - \theta(\cdot,t_*)\|_s + \|\gamma(\cdot,t) - \gamma(\cdot,t_*)\|_{s-3/2} = 0. \tag{101}$$

(Of course, if $t_* = 0$ or $t_* = T$, then the limit in (101) is to be taken as a one-sided limit.) We see in (101) that we are trying to establish convergence in a Hilbert space (i.e., we are showing that $(\theta(\cdot,t), \gamma(\cdot,t))$ converges to $(\theta(\cdot,t_*), \gamma(\cdot,t_*))$ in $H^s \times H^{s-3/2}$, which is a Hilbert space). To establish convergence in a Hilbert space, it is sufficient to establish weak convergence, plus convergence of the norm.

For weak convergence, we focus on θ, but there is no esential difference with γ. Let any s' satisfying $0 < s' < s$ be given. We know that $\theta(\cdot,t) \to \theta(\cdot,t_*)$ in $H^{s'}$. Let $\phi \in H^{-s}$ be given. Since

$0 < s' < s$, we have $-s < -s'$, and thus $H^{-s'}$ is dense in H^{-s}. Since we know that for all $t \in [0,T]$, we have $\theta(\cdot,t) \in H^s$, subject to the uniform bound that comes from the energy estimate, we can let K denote the upper bound; so for any $t \in [0,T]$, we have $\|\theta(\cdot,t)\|_{H^s} \le K$. Let $\delta > 0$ be given. Choose $\phi_\delta \in H^{-s'}$ be given such that $\|\phi - \phi_\delta\|_{H^{-s}} \le \frac{\delta}{3(1+K)}$. Then we compute the following:

$$\langle \theta(\cdot,t) - \theta(\cdot,t_*), \phi\rangle = \langle \theta(\cdot,t) - \theta(\cdot,t_*), \phi_\delta\rangle + \langle \theta(\cdot,t) - \theta(\cdot,t_*), \phi - \phi_\delta\rangle.$$

The second term on the right-hand side is bounded by $\frac{2K\delta}{3(1+K)} \le \frac{2\delta}{3}$. The first integral can be made smaller than $\frac{\delta}{3}$ by taking t sufficiently close to t_*. This proves the weak convergence.

Finally, all that remains is to prove that the $H^s \times H^{s-3/2}$ norm of (θ,γ) is continuous in time. We omit the details of this, as the argument is identical to the corresponding argument in [3]. ∎

6 Uniqueness and continuous dependence

That solutions are unique follows from continuous dependence, so we will actually prove continuous dependence here. We have proved above that solutions (θ,γ) exist in the space $H^s \times H^{s-3/2}$. We therefore will assume that we have two solutions, (θ,γ) and (θ',γ') in this space, and estimate the difference in a lower-regularity space, which we choose as $H^3 \times H^{3/2}$. This space is chosen to be high enough so that the estimates have positive powers of derivatives, but low enough so that the terms can be bounded by $\|\theta\|_{H^s}$ and $\|\gamma\|_{H^{s-3/2}}$.

Before beginning, we note that we will use the estimates in Lemma 7 with unregularized quantities, i.e., for $\varepsilon = 0$. We also remark that the estimate that we now perform, for $(\theta-\theta',\gamma-\gamma')$ in $H^3 \times H^{3/2}$, will be very similar to the energy estimate above for the existence proof.

Theorem 12 *Let $(\theta_0,\gamma_0) \in \mathcal{O}$ and $(\theta_0',\gamma_0') \in \mathcal{O}$ be given, with $\langle\!\langle \sin(\theta_0)\rangle\!\rangle = \langle\!\langle \sin(\theta_0')\rangle\!\rangle = 0$. If there exists $T > 0$ such that there exists $(\theta,\gamma) \in C([0,T];\mathcal{O})$ which solves (25), (31) with $(\theta(\cdot,0),\gamma(\cdot,0)) = (\theta_0,\gamma_0)$, and $(\theta',\gamma') \in C([0,T];\mathcal{O})$ which solves (25), (31) with $(\theta'(\cdot,0),\gamma'(\cdot,0)) = (\theta_0',\gamma_0')$, then there exists $c > 0$ such that*

$$\sup_{t\in[0,T]} \left(|L-L'| + \|\theta-\theta'\|_{H^3} + \|\gamma-\gamma'\|_{H^{3/2}}\right) \le c\left(|L(0)-L'(0)| + \|\theta_0-\theta_0'\|_{H^3} + \|\gamma_0-\gamma_0'\|_{H^{3/2}}\right). \quad (102)$$

Moreover, the solution of the initial value problem (25), (31) with initial data $(\theta_0,\gamma_0) \in \mathcal{O}$ is unique.

Proof: We define E_d to be

$$E_d = Z_0 + Z_1 + Z_2 + Z_3 + Z_4,$$

where

$$Z_0 = \frac{1}{2}(L-L')^2 + \frac{1}{2}\int_0^{2\pi} (\theta-\theta')^2 + (\gamma-\gamma')^2 \, d\alpha,$$

$$Z_1 = \frac{d_1(t)}{2}\int_0^{2\pi} (\partial_\alpha^3\theta - \partial_\alpha^3\theta')^2 \, d\alpha,$$

$$Z_2 = \frac{d_2(t)}{2}\int_0^{2\pi} (\partial_\alpha\gamma - \partial_\alpha\gamma')H(\partial_\alpha^2\gamma - \partial_\alpha^2\gamma') \, d\alpha.$$

The coefficients d_1 and d_2 will be chosen as we make the estimate. We will also define Z_3 and Z_4 in due course. We note, however, that (as in the proof of Theorem 10), all of these choices will be made to ensure that E_d controls a norm; in particular, we will have

$$\frac{1}{2}\left(\|\theta-\theta'\|_{H^3}^2+\|\gamma-\gamma'\|_{H^{3/2}}^2\right)\leq E_d.$$

Before we begin to estimate E_d, we discuss the strategy. We will show that if (θ,γ) and (θ',γ') are in $\mathcal{O}$, then

$$\frac{dE_d}{dt}\leq cE_d.$$

Solving this inequality, we find

$$E_d(t)\leq E_d(0)e^{ct}.$$

This implies both continuous dependence and uniqueness. If we want $E_d(t)$ to be small, then we can achieve this by taking $E_d(0)$ to be sufficiently small. This is for the $H^3\times H^{3/2}$ norm; higher norms follow by interpolation. This is the continuous dependence. For uniqueness, we note that if $E_d(0)=0$, then $E_d(t)=0$, so we have $\theta=\theta'$ and $\gamma=\gamma'$.

It is important to be able to estimate differences of the various quantities associated to (θ,γ) and (θ',γ'), so we will make such estimates now, before beginning the estimate of E_d. The simplest associated quantites are the unit tangent and normal vectors; the following bounds follow from standard Lipschitz estimates for sine and cosine:

$$\|\hat{\mathbf{t}}-\hat{\mathbf{t}}'\|_{H^3}=\|(\cos(\theta)-\cos(\theta'),\sin(\theta)-\sin(\theta'))\|_{H^3}\leq c\|\theta-\theta'\|_{H^3}\leq cE_d^{1/2},$$

and similarly,

$$\|\hat{\mathbf{n}}-\hat{\mathbf{n}}'\|_{H^3}\leq cE_d^{1/2}.$$

Then, since $z_\alpha=\frac{L}{2\pi}\hat{\mathbf{t}}$, a bound for $z_\alpha-z'_\alpha$ follows:

$$\left\|\Phi^{-1}\left(z_\alpha-z'_\alpha\right)\right\|_{H^3}\leq\left\|\left(\frac{L-L'}{2\pi}\right)\hat{\mathbf{t}}\right\|_{H^3}+\frac{L'}{2\pi}\left\|\hat{\mathbf{t}}-\hat{\mathbf{t}}'\right\|_{H^3}\leq cE_d^{1/2}. \tag{103}$$

Next, we estimate $\mathbf{m}-\mathbf{m}'$. We write

$$\Phi(\mathbf{m}-\mathbf{m}')^*=I+II,$$

with the definitions

$$I=z_\alpha\mathcal{K}[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)-z'_\alpha\mathcal{K}[z']\left(\left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right),$$

$$II=\frac{z_\alpha}{2i}\left[H,\frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)-\frac{z'_\alpha}{2i}\left[H,\frac{1}{(z'_\alpha)^2}\right]\left(z'_\alpha\left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right).$$

We rewrite I by adding and subtracting:

$$I=(z_\alpha-z'_\alpha)\mathcal{K}[z_d]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)+z'_\alpha(\mathcal{K}[z_d]-\mathcal{K}[z'])\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right)+z'_\alpha\mathcal{K}[z']\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha-\left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right).$$

With the assumed uniform bounds on θ and γ, and using (103), the first term on the right-hand side is bounded by $cE_d^{1/2}$. For the second term, we apply Lemma 3, and we thus see that this term is also bounded by $cE_d^{1/2}$. For the third term on the right-hand side, its norm in H^3 is bounded by $cE_d^{1/2}$ since $\mathcal{K}[z']$ is a smoothing operator (i.e., we use Lemma 2).

We now turn our attention to the term II. We start by adding and subtracting:

$$II = \frac{z_\alpha - z'_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{z'_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2} - \frac{1}{(z'_\alpha)^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{z'_\alpha}{2i}\left[H, \frac{1}{(z'_\alpha)^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha - z'_\alpha\left(\frac{\gamma'}{z'_\alpha}\right)_\alpha\right).$$

The first term on the right-hand side can be immediately bounded by $cE_d^{1/2}$. The third term can be bounded by $cE_d^{1/2}$ by using Lemma 4. For the second term on the right-hand side, we bound it not by using smoothing properties of commutators, but by not regarding it as a commutator at all. That is, we write the term out, letting the function f temporarily denote $f = z_\alpha(\gamma/z_\alpha)_\alpha$:

$$\left[H, \frac{1}{z_\alpha^2} - \frac{1}{(z'_\alpha)^2}\right]f = H\left(\left(\frac{1}{z_\alpha^2} - \frac{1}{(z'_\alpha)^2}\right)f\right) - \left(\frac{1}{z_\alpha^2} - \frac{1}{(z'_\alpha)^2}\right)Hf.$$

Each of the terms on the right-hand side can clearly be bounded by $cE_d^{1/2}$. This completes the estimate of $\mathbf{m} - \mathbf{m}$; we conclude

$$\|\mathbf{m} - \mathbf{m}'\|_{H^3} \le cE_d^{1/2}. \tag{104}$$

We next estimate $V_W - V'_W$. From formula (24), after adding and subtracting several times, and using the above estimates for $\hat{\mathbf{t}} - \hat{\mathbf{t}}'$ and $\mathbf{m} - \mathbf{m}'$, the following estimate can be found:

$$\|V_W - V'_W\|_{H^2} \le cE_d^{1/2}. \tag{105}$$

We omit the details of most other differences to be estimated, since they are similar to the above. However, we do remark that one interesting estimate is for the difference $(I + 2A\mathcal{J}[z_d])^{-1} - (I + 2A\mathcal{J}[z'_d])^{-1}$. We consider invertible linear operators B_1 and B_2, and we see that

$$B_1^{-1} - B_2^{-1} = B_1^{-1}B_2B_2^{-1} - B_1^{-1}B_1B_2^{-1} = B_1^{-1}(B_2 - B_1)B_2^{-1}. \tag{106}$$

Thus, we see that we can make a Lipschitz estimate for the inverses by estimating the inverses individually and by making a Lipschitz estimate for the forward operators. If we let $B_1 = I + 2A\mathcal{J}[z_d]$ and $B_2 = I + 2A\mathcal{J}[z'_d]$, then B_1 and B_2 are bounded invertible operators by Lemma 6, and we see from (106) that we need to be able to estimate $B_2 - B_1 = 2A(\mathcal{J}[z'_d] - \mathcal{J}[z_d])$. Since we can decompose $\mathcal{J}$ as commutators plus terms involving $\mathcal{K}$, and since we know how to make Lipschitz estimates for both of these kinds of terms, we are able to make the estimate for the difference.

We can use these estimates to conclude

$$\frac{dZ_0}{dt} \le cE_d. \tag{107}$$

Now, we get to the most important part of the proof of the theorem, which is to take the time derivative of Z_1 and Z_2. We have

$$\frac{dZ_1}{dt} = d_1\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')(\partial_\alpha^3\theta_t - \partial_\alpha^3\theta'_t)\ d\alpha + \frac{dd_1}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')^2\ d\alpha. \tag{108}$$

Applying ∂_α^3 to (25), we get

$$\partial_\alpha^3\theta_t = \frac{2\pi^2}{L^2}H\partial_\alpha^4\gamma + \frac{2\pi}{L}(\partial_\alpha^3 V_W)\theta_\alpha + \frac{2\pi}{L}V_W(\partial_\alpha^4\theta) + Y_1,$$

where Y_1 is defined as

$$Y_1 = \frac{6\pi}{L}(\partial_\alpha^2 V_W)(\partial_\alpha^2\theta) + \frac{6\pi}{L}(\partial_\alpha V_W)(\partial_\alpha^3\theta) + \frac{2\pi}{L}\partial_\alpha^3(\mathbf{m}\cdot\hat{\mathbf{n}}).$$

Of course, we get the corresponding formula for $\partial_\alpha^3\theta'$. These considerations allow us to substitute into (108):

$$\frac{dZ_1}{dt} = d_1\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')\left(\frac{2\pi^2}{L^2}H\partial_\alpha^4\gamma - \frac{2\pi^2}{(L')^2}H\partial_\alpha^4\gamma'\right)\,d\alpha$$
$$+d_1\int_0^{2\pi}(\partial_\alpha^3\theta-\partial_\alpha^3\theta')\left(\frac{2\pi}{L}(\partial_\alpha^3 V_W)\theta_\alpha - \frac{2\pi}{L'}(\partial_\alpha^3 V_W')\theta_\alpha'\right)\,d\alpha + d_1\int_0^{2\pi}(\partial_\alpha^3\theta-\partial_\alpha^3\theta')\left(\frac{2\pi}{L}V_W(\partial_\alpha^4\theta) - \frac{2\pi}{L'}V_W'(\partial_\alpha^4\theta')\right)\,d\alpha$$
$$+ d_1\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')(Y_1 - Y_1')\,d\alpha + \frac{dd_1}{dt}\cdot\frac{1}{2}\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')^2\,d\alpha. \quad (109)$$

There are five integrals on the right-hand side of (109), and we give these names; we write

$$\frac{dZ_1}{dt} = \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_5.$$

We must manipulate and estimate these, and we begin now with Υ_1.

We add and subtract to write Υ_1 as follows:

$$\Upsilon_1 = d_1\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')\left(\frac{2\pi^2}{L^2}\right)(H\partial_\alpha^4\gamma - H\partial_\alpha^4\gamma')\,d\alpha + d_1\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')\left(\frac{2\pi^2}{L^2} - \frac{2\pi^2}{(L')^2}\right)H\partial_\alpha^4\gamma'\,d\alpha.$$

Of the two integrals on the right-hand side, the second can clearly be bounded in terms of E_d since the definition of Z_0 includes $(L - L')^2$ (and recall that L and L' must both be greater than or equal to 2π). Therefore, we have

$$\Upsilon_1 \le d_1\int_0^{2\pi}(\partial_\alpha^3\theta - \partial_\alpha^3\theta')\left(\frac{2\pi^2}{L^2}\right)(H\partial_\alpha^4\gamma - H\partial_\alpha^4\gamma')\,d\alpha + cE_d.$$

To work with Υ_2, we first expand $\partial_\alpha^3 V_W$:

$$\partial_\alpha^3 V_W = \frac{\pi}{L}\theta_\alpha H\gamma_{\alpha\alpha} + \frac{\pi}{L}[H,\theta_\alpha](\gamma_{\alpha\alpha}) + \frac{2\pi}{L}H(\gamma_\alpha\theta_{\alpha\alpha}) + \frac{\pi}{L}H(\gamma\partial_\alpha^3\theta) - \partial_\alpha^2(\mathbf{m}\cdot\hat{\mathbf{t}}).$$

After some adding and subtracting, the leading-order term of

$$\frac{2\pi}{L}(\partial_\alpha^3 V_W)\theta_\alpha - \frac{2\pi}{L'}(\partial_\alpha^3 V_W')\theta_\alpha'$$

can then be seen to be

$$\frac{2\pi^2}{L^2}\theta_\alpha^2(H\gamma_{\alpha\alpha} - H\gamma_{\alpha\alpha}');$$

indeed, the contributions to Υ_2 from all the other terms can then be bounded in terms of E_d. These considerations yield the following:

$$\Upsilon_2 \leq d_1 \int_0^{2\pi} (\partial_\alpha^3\theta - \partial_\alpha^3\theta') \left(\frac{2\pi^2}{L^2}\theta_\alpha^2\right) (H\partial_\alpha^2\gamma - H\partial_\alpha^2\gamma')\ d\alpha + cE_d.$$

Each of Υ_3, Υ_4, and Υ_5 are bounded by cE_d. For Υ_4, this follows from the estimates such as (104) and (105). For Υ_5, the fact that it can be bounded by cE_d follows immediately since $\frac{dd_1}{dt}$ is bounded; of course, we have not yet defined d_1, but once we do, it will be clear that d_1 and $\frac{dd_1}{dt}$ are bounded. To be slightly more precise, d_1, when we do define it, will be in terms of (θ, γ) only, and we have assumed that (θ, γ) is a bounded solution.

For Υ_3, we add and subtract to reach the following:

$$\Upsilon_3 = d_1 \int_0^{2\pi} (\partial_\alpha^3\theta - \partial_\alpha^3\theta') \left(\frac{2\pi}{L}V_W\right) (\partial_\alpha^4\theta - \partial_\alpha^4\theta')\ d\alpha + d_1 \int_0^{2\pi} (\partial_\alpha^3\theta - \partial_\alpha^3\theta') \left(\frac{2\pi}{L}V_W - \frac{2\pi}{L'}V_W'\right) (\partial_\alpha^4\theta')\ d\alpha.$$

The first of these can be integrated by parts:

$$\Upsilon_3 = \frac{d_1}{2} \int_0^{2\pi} \left(\frac{2\pi}{L}\partial_\alpha V_W\right) (\partial_\alpha^3\theta - \partial_\alpha\theta')^2\ d\alpha. + d_1 \int_0^{2\pi} (\partial_\alpha^3\theta - \partial_\alpha^3\theta') \left(\frac{2\pi}{L}V_W - \frac{2\pi}{L'}V_W'\right) (\partial_\alpha^4\theta')\ d\alpha.$$

Both of these terms, then, can be bounded by cE_d. So far, then, we have calculated

$$\begin{aligned}\frac{dZ_1}{dt} &= \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_5 \leq cE_d \\ &+ \int_0^{2\pi} \left(\frac{2\pi^2 d_1}{L^2}\right) (\partial_\alpha^3(\theta - \theta'))H\partial_\alpha^4(\gamma - \gamma')\ d\alpha + \int_0^{2\pi} \left(\frac{2\pi^2 d_1\theta_\alpha^2}{L^2}\right) (\partial_\alpha^3(\theta - \theta'))H\partial_\alpha^2(\gamma - \gamma')\ d\alpha.\end{aligned}$$

We now are ready to take the time derivative of Z_2. To begin, we have

$$\frac{dZ_2}{dt} = d_2 \int_0^{2\pi} (\gamma_{\alpha t} - \gamma_{\alpha t}')(H\partial_\alpha^2\gamma - H\partial_\alpha^2\gamma')\ d\alpha + \frac{dd_2}{dt} \cdot \frac{1}{2} \int_0^{2\pi} (\gamma_\alpha - \gamma_\alpha')(H\partial_\alpha^2\gamma - H\partial_\alpha^2\gamma')\ d\alpha.$$

We take the derivative of γ_t, finding

$$\partial_\alpha\gamma_t = -\left(\frac{8\pi^3 S}{L^3}\right)\partial_\alpha^5\theta - \left(\frac{8\pi^3 S}{L^3}\right)(\partial_\alpha^3\theta)\left(\frac{3}{2}\theta_\alpha^2 - \frac{L^2\bar{c}_1}{4\pi^2}\right) + \gamma_{\alpha\alpha}Q_2 + Y_2, \tag{110}$$

where Y_2 denotes the following collection of terms:

$$Y_2 = -\widetilde{S}\theta_{\alpha\alpha}\partial_\alpha Q_1 + \gamma_\alpha\partial_\alpha Q_2 + \partial_\alpha Q_3.$$

We dispense with many of the details; we only get two significant terms from $\frac{dZ_2}{dt}$, and these correspond to the first two terms on the right-hand side of (110). We get the following estimate:

$$\begin{aligned}\frac{dZ_2}{dt} \leq cE_d &+ \int_0^{2\pi} \left(-\frac{8\pi^3 S d_2}{L^3}\right) (\partial_\alpha^5(\theta - \theta'))(H\partial_\alpha^2(\gamma - \gamma'))\ d\alpha \\ &+ \int_0^{2\pi} \left(-\frac{12\pi^3 S d_2\theta_\alpha^2}{L^3} + \frac{2\pi S d_2\bar{c}_1}{L}\right) (\partial_\alpha^3(\theta - \theta'))(H\partial_\alpha^2(\gamma - \gamma'))\ d\alpha.\end{aligned} \tag{111}$$

We choose d_1 and d_2 so that

$$\frac{2\pi^2 d_1}{L^2} - \frac{8\pi^3 S d_2}{L^3} = 0.$$

Specifically, we make the choices

$$d_2 = \max\left\{1, \frac{1}{S}\right\}, \qquad d_1 = \frac{4\pi S d_2}{L}.$$

Then, when adding $\frac{dZ_1}{dt}$ and $\frac{dZ_2}{dt}$, the leading terms cancel, leaving the following:

$$\frac{dZ_1}{dt} + \frac{dZ_2}{dt} \leq cE_d + \int_0^{2\pi} \left(\frac{2\pi^2 d_1 \theta_\alpha^2}{L^2} - \frac{12\pi^3 S d_2 \theta_\alpha^2}{L^3} + \frac{2\pi S d_2 \bar{c}_1}{L}\right)(\partial_\alpha^3(\theta - \theta'))(H\partial_\alpha^2(\gamma - \gamma'))\ d\alpha. \quad (112)$$

Using the definition of d_1, this simplifies:

$$\frac{dZ_1}{dt} + \frac{dZ_2}{dt} \leq cE_d + \int_0^{2\pi} \left(-\frac{4\pi^3 S d_2 \theta_\alpha^2}{L^3} + \frac{2\pi S d_2 \bar{c}_1}{L}\right)(\partial_\alpha^3(\theta - \theta'))(H\partial_\alpha^2(\gamma - \gamma'))\ d\alpha. \quad (113)$$

We now define Z_3, and we will soon define Z_4; these terms will allow us to cancel the integral on the right-hand side of (113); notice that this integral is not bounded in terms of E_d, since it has too many derivatives. We let Z_3 be given by

$$Z_3 = \frac{d_3(t)}{2} \int_0^{2\pi} (\partial_\alpha^2 \theta - \partial_\alpha^2 \theta')^2\ d\alpha.$$

Taking its time derivative, we get

$$\frac{dZ_3}{dt} = d_3 \int_0^{2\pi} (\partial_\alpha^2 \theta_t - \partial_\alpha^2 \theta_t')(\partial_\alpha^2 \theta - \partial_\alpha^2 \theta')\ d\alpha + \frac{dd_3}{dt} \cdot \frac{1}{2} \int_0^{2\pi} (\partial_\alpha^2 \theta - \partial_\alpha^2 \theta')\ d\alpha.$$

Applying two spatial derivatives to θ_t, we get the following:

$$\partial_\alpha^2 \theta_t = \frac{2\pi^2}{L^2} H\partial_\alpha^3 \gamma + Y_3,$$

where the term Y_3 is defined as

$$Y_3 = \partial_\alpha^2 \left(\frac{2\pi}{L} V_W \theta_\alpha + \frac{2\pi}{L} \mathbf{m} \cdot \hat{\mathbf{n}}\right). \quad (114)$$

As above, we do not include all details, but there is only one term which is not bounded in terms of E_d :

$$\frac{dZ_3}{dt} \leq cE_d + \int_0^{2\pi} \left(\frac{2\pi^2 d_3}{L^2}\right)(H\partial_\alpha^3(\gamma - \gamma'))(\partial_\alpha^2(\theta - \theta'))\ d\alpha.$$

In the integral on the right-hand side, we integrate by parts once:

$$\frac{dZ_3}{dt} \leq cE_d + \int_0^{2\pi} \left(-\frac{2\pi^2 d_3}{L^2}\right)(H\partial_\alpha^2(\gamma - \gamma'))(\partial_\alpha^3(\theta - \theta'))\ d\alpha. \quad (115)$$

For the final piece of E_d, we define Z_4. Before doing so, we note that the coefficients d_1, d_2, and d_3 that we have introduced thus far are functions of t only. Now, however, the term Z_4 will involve a coefficient d_4 which we must take to depend on both α and t; this is because it will be used to cancel terms in (112) which involve θ_α^2. With this in mind, our definition is

$$Z_4 = \frac{1}{2}\int_0^{2\pi} \left(\sqrt{d_4(\alpha,t)}(\gamma-\gamma')\right) H\partial_\alpha \left(\sqrt{d_4(\alpha,t)}(\gamma-\gamma')\right)\ d\alpha.$$

We note that the form of Z_4 may at first look unusual; we note, however, that it is in the form $\int g\Lambda g\ d\alpha$, with $g = \sqrt{d_4}(\gamma-\gamma')$, and we may thus use (1) to estimate its time derivative. Furthermore, we will choose d_4 to satisfy $d_4 \geq 1$ so that it is clear that derivatives of $\sqrt{d_4}$ are bounded. Taking the time derivative of Z_4, we get

$$\frac{dZ_4}{dt} = \int_0^{2\pi} \sqrt{d_4}(\gamma_t-\gamma_t')H\partial_\alpha\left(\sqrt{d_4}(\gamma-\gamma')\right)\ d\alpha + \int_0^{2\pi}\frac{dd_4}{dt}\cdot\frac{1}{2\sqrt{d_4}}(\gamma-\gamma')H\partial_\alpha\left(\sqrt{d_4}(\gamma-\gamma')\right)\ d\alpha.$$

As for Z_3, when considering Z_4, there is only one significant term, and this is from the leading-order term in (31). After our usual adding and subtracting, we arrive at

$$\frac{dZ_4}{dt} \leq cE_d + \int_0^{2\pi}\left(-\frac{8\pi^3 S}{L^3}\right)\sqrt{d_4}(\partial_\alpha^4(\theta-\theta'))H\partial_\alpha(\sqrt{d_4}(\gamma-\gamma'))\ d\alpha.$$

We are able to pass the second factor of $\sqrt{d_4}$ above through $H\partial_\alpha$, incurring only terms which can be bounded by cE_d. We are left, then with

$$\frac{dZ_4}{dt} \leq cE_d + \int_0^{2\pi}\left(-\frac{8\pi^3 S d_4}{L^3}\right)(\partial_\alpha^4(\theta-\theta'))H\partial_\alpha(\gamma-\gamma')\ d\alpha.$$

We integrate by parts in this integral, retaining just one integral that is not bounded by cE_d :

$$\frac{dZ_4}{dt} \leq cE_d + \int_0^{2\pi}\left(\frac{8\pi^3 S d_4}{L^3}\right)(\partial_\alpha^3(\theta-\theta'))H\partial_\alpha^2(\gamma-\gamma')\ d\alpha. \tag{116}$$

We now add (113), (115), and (116), arriving at the following:

$$\begin{aligned}\frac{d(Z_1+Z_2+Z_3+Z_4)}{dt} &\leq cE_d \\ &+ \int_0^{2\pi}\left(-\frac{4\pi^3 S d_2\theta_\alpha^2}{L^3} + \frac{2\pi S d_2 \bar{c}_1}{L} - \frac{2\pi^2 d_3}{L^2} + \frac{8\pi^3 S d_4}{L^3}\right)(\partial_\alpha^3(\theta-\theta'))H\partial_\alpha^2(\gamma-\gamma')\ d\alpha. \end{aligned}\tag{117}$$

The choice of d_3 and d_4 depends on the sign of $\bar{c}_1$; this is exactly the same as occured when choosing c_3 and c_4 in the proof of Theorem 10. Indeed, our choices of d_3 and d_4 are almost identical:

$$d_3 = \frac{LSd_2(\bar{c}_1)^+}{\pi} + \frac{4\pi S}{L},$$

$$d_4 = \frac{L^2 d_2(\bar{c}_1)^-}{4\pi^2} + \frac{d_2}{2}\theta_\alpha^2 + 1.$$

With these choices, then, the integral on the right-hand side of (117) vanishes. Combining (117) with (107), we conclude

$$\frac{dE_d}{dt} \leq cE_d.$$

As discussed at the beginning of the proof, this completes the proof. ∎

7 Conclusion

We have presented a well-posedness theory for the initial value problem describing the evolution of hydroelastic waves in two dimensions. Our model assumes a thin, massless elastic sheet interacts with an inviscid, irrotational flow. The elastic model accounts for membrane bending stresses and surface tension. We prove short time well-posedness in Sobolev spaces. More precisely, given periodic initial data $\theta(\cdot,) \in H^s$ and $\gamma(\cdot, 0) \in H^{s-3/2}$ for s large enough so that our estimates hold, there is a nonzero time in which the solution exists, is unique, has the same regularity as the initial conditions, and depends continuously on the data. The proof is based on energy estimates, and makes use of an arclength-angle representation of the interface and a small scale decomposition first introduced for computational reasons in [25].

In future work, we expect to treat the three-dimensional problem, as well as the problem with mass (in either two or three dimensions). To replace the arclength parameterization, we expect to use a generalized isothermal parameterization as discussed in [8], [9].

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8 Appendix: Pressure jump at the interface

The elastic interface deforms due to the pressure exterted on it by the fluids. In this section, the relation (19) for the pressure jump at the interface is derived. The approach follows [12].

We imagine the 1D interface $(x(\alpha, t), y(\alpha, t))$ to be the trace in the $x - y$ plane of a 2D elastic sheet or plate with no variation in the $z-$direction. Suppressing the dependence on time, let $\mathbf{F}(s)$ be the resultant internal force and $\mathbf{M}(s)$ the moment of internal force (both per unit length in the $z-$direction) on a cross section of the sheet, which is assumed to have thickness h. The resultant external force on the sheet is the jump in pressure at the interface, $[p] = (p_1 - p_2)|_S$. The equations of mechanical equilibrium for the deformed sheet or plate are [22], [30]

$$\frac{d\mathbf{F}}{ds} = -[p]\hat{\mathbf{n}}, \qquad \frac{d\mathbf{M}}{ds} = \mathbf{F} \times \hat{\mathbf{t}}. \tag{118}$$

Decompose $\mathbf{F}(s)$ into tangential and normal components as $\mathbf{F}(s) = T(s)\hat{\mathbf{t}} + N(s)\hat{\mathbf{n}}$, and note that $\mathbf{M}(s)$ is in the $z-$direction so we can write $\mathbf{M}(s) = M(s)\hat{\mathbf{k}}$. Substitute these relations into (118), take the derivative with respect to s and use the Frenet formulae to obtain

$$\begin{aligned} (T' - N\kappa)\hat{\mathbf{t}} + (N' + \kappa T)\hat{\mathbf{n}} &= -[p]\hat{\mathbf{n}}, \\ M' &= -N, \end{aligned} \tag{119}$$

where the prime denotes derivative with respect to s. Equating normal and tangential components in (119), we obtain the system of equations

$$\begin{aligned} T' - N\kappa &= 0, \\ N' + \kappa T &= -[p], \\ M' + N &= 0. \end{aligned} \tag{120}$$

We assume a linear constitutive relationship for the elastic moment [34]

$$M = E_B \kappa \tag{121}$$

where E_B is the bending modulus. Thin shell theory provides the relation $E_B = Eh^3/[12(1-\nu^2)]$, where E is Young's modulus, ν is Poisson's ratio and h is the plate thickness. Insert (121) into the third equation of (120) to eliminate M and N and integrate once with respect to arclength to obtain the relation for the pressure jump

$$[p] = E_b \kappa'' + \frac{E_B}{2}\kappa^3 - c_1(t)\kappa. \tag{122}$$

where $c_1(t)$ is a constant (in space) of integration. The first two terms on the right hand side of (19) represent the internal bending stress of the elastic sheet, and the thrid term is surface tension. This result is consistent with the hydroelastic model of Plotnikov and Toland [32].

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