VORTEX SHEETS, BOUSSINESQ EQUATIONS, AND OTHER PROBLEMS IN THE WIENER ALGEBRA

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Abstract. The vortex sheet is a problem in interfacial fluid dynamics which is known to have an ill-posed initial value problem in Sobolev spaces. We sketch a proof of this ill-posedness that follows from an existence theorem for vortex sheets by Duchon and Robert, in which Duchon and Robert constructed vortex sheet solutions on the stable manifold using a fixed point formulation in function spaces related to the Wiener algebra. We give a detailed exposition of those parts of the Duchon-Robert proof which can be applied to other problems, and give an overview of some other such applications. These other applications include demonstrating nonlinear ill-posedness of some linearly ill-posed Boussinesq equations, and developing existence theory for problems in epitaxial growth of thin films and mean field games.

1. Introduction

A fundamental question for systems of differential equations is well-posedness in the sense of Hadamard: do solutions exist, are the solutions unique, and do solutions depend continuously on the initial data? Well-posedness of a system speaks to its predictability and suitability for simulation: if for example solutions of a model do not depend continuously on the data, then it would be difficult to impossible to generate predictions from the model to verify in a laboratory or computational setting. While there are widely-used methods to prove well-posedness of systems of differential equations, such as the energy method [45], proving ill-posedness can at times seem more elusive at least for nonlinear equations.

In many cases the question of well-posedness or ill-posedness is more clear for linear equations, so a standard first step to understand well-posedness of nonlinear equations can be to linearize the system in question and then employ techniques such as Fourier analysis. Linear theory is not, however, perfectly predictive of nonlinear behavior; Beale and Schaeffer give an example of an equation which is linearly ill-posed but nonlinearly well-posed [10]. We will present an analytical method which has been used to demonstrate that some systems which are linearly ill-posed are in fact nonlinearly ill-posed; this method is useful when the ill-posedness stems from ellipticity of the linear system in space-time. This method was introduced by Duchon and Robert in proving existence of some solutions for the vortex sheet problem [20].

The unregularized vortex sheet is a classical problem in fluid dynamics in which one fluid shears past the other. More specifically, two fluids are separated by a sharp interface, and the normal component of the fluid velocities must match at the interface; the interface then moves according to this normal velocity. The tangential velocities generally do not match. When calculating the vorticity, which is the curl of the velocity, since the velocity in the combined fluid region is discontinuous, the vorticity is measure-valued, i.e. the vorticity is an amplitude multiplied by a Dirac mass supported on the interface between the two fluids. This amplitude is known as the vortex sheet strength, and a closed system of evolution equations can be formed for this vortex sheet strength and the position of the interface.

These evolution equations for the vortex sheet are famously known to have an ill-posed initial value problem (in Sobolev spaces). Proofs of this ill-posedness include [16], [29], [35], [47]. The vortex sheet problem does, however, have analytic solutions [46]. A main focus of the present work is to present a result by Duchon and Robert on the vortex sheet which combines these aspects, existence of analytic solutions and ill-posedness in Sobolev spaces [20]. We note that the vortex sheet exhibits what is known as the Kelvin-Helmholtz instability, and this instability is the same thing as the ill-posedness of the initial value problem. This need not always be the case; a problem could be well-posed but solutions might exhibit growth, giving rise to instability. But in the case of the vortex sheet, the instability is so strong as to lead to ill-posedness.

Our aim in this article is to give a detailed exposition of those aspects of the Duchon-Robert proof which are applicable to other problems, while treating those aspects more specific to the vortex sheet somewhat more briefly. We will then demonstrate several other problems to which the method may be applied. We give this exposition of the Duchon-Robert result in Section 2. Another elliptic problem, a family of Boussinesq equations, is treated in Section 3. We discuss applications of the method to parabolic problems in Section 4, describing some specific parabolic problems in Section 4.1 and 4.2. We offer concluding remarks in Section 5.

2. The Duchon-Robert vortex sheet result

We now describe the results of Duchon and Robert on existence of vortex sheets, and the consequences of these results for ill-posedness [20].

2.1. **Function spaces.**

Given $\rho \geq 0$, we define the function space B_ρ to be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that the following norm is finite:

$$\|f\|_{B_\rho} = \int_{\mathbb{R}} e^{\rho|\xi|}|\hat{f}(\xi)|\ d\xi.$$

When $\rho = 0$, this space is the Wiener algebra. For $\rho > 0$, a function $f \in B_\rho$ is analytic with radius of analyticity at least ρ. We have the algebra property for B_ρ, which we now demonstrate. Let f and g each be in B_ρ; we may then bound the norm of the product:

$$\|fg\|_{B_\rho} \leq \int_{\mathbb{R}} e^{\rho|\xi|}|\widehat{fg}(\xi)|\ d\xi \leq \int_{\mathbb{R}} e^{\rho|\xi-\eta|}|\hat{f}(\xi-\eta)|e^{\rho|\eta|}|\hat{g}(\xi)|\ d\eta d\xi \leq \|f\|_{B_\rho}\|g\|_{B_\rho}.$$

Letting $\alpha > 0$ be given, we define a space-time version of these spaces related to the Wiener algebra. The space $\mathcal{B}_\alpha$ is the set of all $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, continuous in time, such that the following norm is finite:

$$\|f\|_{\mathcal{B}_\alpha} = \int_{\mathbb{R}} \left(\sup_{t\in[0,\infty)} e^{\alpha t|\xi|}|\hat{f}(t,\xi)| \right)\ d\xi.$$

If $f \in \mathcal{B}_\alpha$, then for any $t \geq 0$, we have $f(t,\cdot) \in B_{\alpha t}$. Thus if $f \in \mathcal{B}_\alpha$, for any $t > 0$, we see that $f(t,\cdot)$ is analytic with radius of analyticity at least αt. A similar computation as for B_ρ demonstrates that $\mathcal{B}_\alpha$ also inherits the algebra property of the Wiener algebra,

$$\|fg\|_{\mathcal{B}_\alpha} \leq \|f\|_{\mathcal{B}_\alpha}\|g\|_{\mathcal{B}_\alpha}.$$

2.2. **Operators and estimates.** We now introduce a few operators on these function spaces. These are operators which arise in the Duhamel formula for the vortex sheet, and are the semigroup for the linear problem and the Duhamel integral operators which will eventually be applied to the nonlinear terms in the vortex sheet problem. While we introduce the operators now, we will put them together into the Duhamel formula we use in the subsequent section.

The operator Λ, acting on functions $f:\mathbb{R}\to\mathbb{R}$, is the Fourier multiplier operator given by

$$\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi).$$

There are a number of other equivalent descriptions of Λ, for instance we could say $\Lambda = H\partial_x$ where H is the Hilbert transform, or we could say $\Lambda = \sqrt{-\partial_{xx}}$.

We introduce notation for our semigroup, $S(t) = e^{-t\Lambda}$. We can show that for $\alpha\in(0,1)$, $S: B_0 \to \mathcal{B}_\alpha$ is a bounded linear operator. Let $f\in B_0$ be given; then we calculate the norm of Sf :

$$\|Sf\|_{\mathcal{B}_\alpha} = \int_{\mathbb{R}} \sup_{t\in[0,\infty)} e^{\alpha t|\xi|}e^{-t|\xi|}|\hat{f}(\xi)|\ d\xi.$$

The supremum here satisfies

$$\sup_{t\in[0,\infty)} e^{t|\xi|(\alpha-1)} = 1,$$

for $\alpha<1$. We therefore have exactly that $\|Sf\|_{\mathcal{B}_\alpha} = \|f\|_{B_0}$.

We define the operators I^+ and I^-, acting on functions in $\mathcal{B}_\alpha$:

$$(I^+h)(t,\cdot) = \int_0^t e^{(s-t)\Lambda}h_x(s,\cdot)\ ds,$$

$$(I^-h)(t,\cdot) = \int_t^\infty e^{(t-s)\Lambda}h_x(s,\cdot)\ ds.$$

We can show that these operators are bounded from $\mathcal{B}_\alpha$ to $\mathcal{B}_\alpha$. We begin calculating the norm of I^+h, for any $h\in\mathcal{B}_\alpha$:

$$\begin{aligned}\|I^+h\|_{\mathcal{B}_\alpha} &= \int_{\mathbb{R}} \left(\sup_{t\in[0,\infty)} e^{\alpha t|\xi|}|\widehat{I^+h}(t,\xi)|\right)\ d\xi \\ &\leq \int_{\mathbb{R}}\left(\sup_{t\in[0,\infty)} e^{\alpha t|\xi|}\int_0^t e^{(s-t)|\xi|}|\xi||\hat{h}(s,\xi)|\ ds\right)\ d\xi.\end{aligned}$$

We manipulate the exponentials, and pull the transform of h through the temporal integral by bounding it with its supremum:

$$\begin{aligned}&\|I^+h\|_{\mathcal{B}_\alpha} \\ &\quad\leq \int_{\mathbb{R}}\left(|\xi|\sup_{t\in[0,\infty)} e^{\alpha t|\xi|-t|\xi|}\int_0^t e^{s|\xi|-\alpha s|\xi|}\left(\sup_{\tau\in[0,\infty)} e^{\alpha\tau|\xi|}|\hat{h}(\tau,\xi)|\right)\ ds\right)\ d\xi \\ &\quad\leq \|h\|_{\mathcal{B}_\alpha}\left(\sup_{\xi\in\mathbb{R}}\sup_{t\in[0,\infty)}|\xi|e^{\alpha t|\xi|-t|\xi|}\int_0^t e^{s|\xi|-\alpha s|\xi|}\ ds\right).\end{aligned}$$

We can calculate the quantity which multiplies $\|h\|_{\mathcal{B}_\alpha}$ here:

$$\sup_{\xi\in\mathbb{R}}\sup_{t\in[0,\infty)}|\xi|e^{\alpha t|\xi|-t|\xi|}\int_0^t e^{s|\xi|-\alpha s|\xi|}\ ds$$
$$=\sup_{\xi\in\mathbb{R}}\sup_{t\in[0,\infty)}|\xi|e^{\alpha t|\xi|-t|\xi|}\left(\frac{e^{t|\xi|-\alpha t|\xi|}-1}{|\xi|-\alpha|\xi|}\right)=\sup_{\xi\in\mathbb{R}}\sup_{t\in[0,\infty)}\frac{1-e^{t|\xi|(\alpha-1)}}{1-\alpha}\leq\frac{1}{1-\alpha},$$

if $\alpha<1$. We conclude that the operator norm of I^+ satisfies

$$\|I^+\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}\leq\frac{1}{1-\alpha}.$$

Analagous calculations for I^- may be carried out, yielding

$$\|I^-\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}\leq\frac{1}{1+\alpha}.$$

We will need a special case of the operator I^-, which is $I_0=I^-(0)$. That is, given $h\in\mathcal{B}_\alpha$, we define $I_0h=(I^-h)(0,\cdot)\in B_0$. We will need to estimate SI_0, so we compute as follows:

$$\|SI_0h\|_{\mathcal{B}_\alpha}=\|I_0h\|_{B_0}=\int_{\mathbb{R}}\widehat{I_0h}(\xi)\ d\xi=\int_{\mathbb{R}}\left|\int_0^\infty e^{-s|\xi|}\hat{h}(s,\xi)\ ds\right|\ d\xi.$$

We use the triangle inequaility and replace the integral over $(0,\infty)$ with a supremum of integrals over (t,∞) :

$$\|SI_0h\|_{\mathcal{B}_\alpha}\leq\int_{\mathbb{R}}\sup_{t\in[0,\infty)}e^{\alpha t|\xi|}\int_t^\infty e^{(t-s)|\xi|}|\hat{h}(s,\xi)|\ dsd\xi.$$

We then argue as in the bound for the operator norm of I^+, concluding

$$\|SI_0\|_{\mathcal{B}_\alpha\to\mathcal{B}_\alpha}\leq\frac{1}{1+\alpha}.$$

2.3. **The vortex sheet.** The vortex sheet is the interface between two fluids, each of which satisfy the incompressible, irrotational Euler equations. Each fluid may have its own constant density, and in the present case we take these densities to be equal to each other. We take one fluid to be above the other, and in the present case we take the interface to be asymptotic to the real line at horizontal infinity. A simple schematic of this situation is shown in Figure 1.

Taking the limit of the fluid velocities as the interface is approached, the normal components must be equal when approaching from either above or below. This means that the interface moves according to this normal velocity. There may be, however, and there typically is, a jump in the tangential velocity across the interface. When calculating the vorticity, which is equal to the curl of the velocity, in the entire fluid region, then, there is a Dirac mass (that is, the curl is a derivative operator, and when applied to the velocity, which has a jump, a Dirac mass arises). More specifically, the vorticity is equal to an amplitude times the Dirac mass; this amplitude varies along the interface, and is known as the vortex sheet strength. The unknowns in the vortex sheet system are the position of the interface and the vortex sheet strength; these together entirely specify the vorticity, and the fluid velocity everywhere may be recovered from the vorticity via the Biot-Savart law.

We parameterize the vortex sheet by horizontal position, so that the curve is given by $(x,y(x))$. We define the variable $v=y_x$. We write the vortex sheet strength,

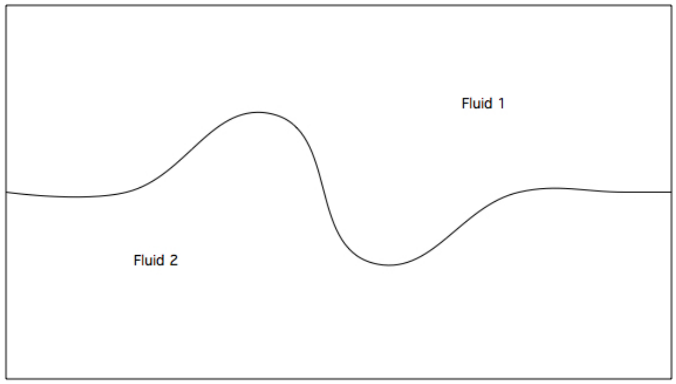

FIGURE 1. A schematic of a vortex sheet.

Ω, as $\Omega = 1 + \omega$, where have chosen Ω to have mean one and for ω to have zero mean; the mean vortex sheet strength is conserved by the equations of motion.

The equations of motion for the vortex sheet, in the form expressed by Duchon and Robert, are

$$v_t - \Lambda\omega = (F(v,\omega))_x, \tag{1}$$

$$\omega_t - \Lambda v = (G(v,\omega))_x. \tag{2}$$

We have mentioned already that these equations are elliptic; this can be seen from 1, 2. Linearizing the system just means setting F and G equal to zero; then the system is just a factored version of Laplace's equation in space-time. That is, if $F = G = 0$, we have $v_{tt} = \Lambda\omega_t = \Lambda^2 v = -v_{xx}$.

Although we will not dwell on the precise form of the nonlinearities F and G for the full vortex sheet problem, we do note that they are given by the formulas

$$\begin{aligned} F(t,x) = \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \left(\frac{1}{1+p^2} - 1\right)\left(\frac{1+\omega(t,x')}{x-x'}\right)\,dx' \\ + \frac{1}{\pi}v(t,x)\mathrm{PV}\int_{\mathbb{R}}\left(\frac{p}{1+p^2}\right)\left(\frac{1+\omega(t,x'))}{x-x'}\right)\,dx', \end{aligned}$$

$$\begin{aligned} G(t,x) = \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\left(\frac{p}{1+p^2} - p\right)\frac{dx'}{x-x'} \\ + \frac{1}{\pi}\omega(t,x)\mathrm{PV}\int_{\mathbb{R}}\left(\frac{p}{1+p^2}\right)\left(\frac{1+\omega(t,x')}{x-x'}\right)\,dx' \\ + \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\left(\frac{p}{1+p^2}\right)\left(\frac{\omega(t,x')}{x-x'}\right)\,dx', \end{aligned}$$

where these are principal value integrals in the Cauchy sense, and where the quantity p represents

$$p = p(t,x,x') = \frac{y(t,x) - y(t,x')}{x-x'}.$$

Note that although we have written p in terms of y, it can be represented just as well in terms of $v = y_x$ through the divided difference formula

$$\frac{y(t,x) - y(t,x')}{x - x'} = \int_0^1 y_x(t, \alpha x + (1-\alpha)x')\ d\alpha.$$

The interested reader might consult [9], [15], [37], [44] for a derivation of the equations of motion for the vortex sheet. Briefly stated, the derivation starts with the form of the vorticity we have described above, that the vorticity is equal to Ω, the vortex sheet strength, multiplied by the Dirac mass along the interface. Then the velocity in the interior of the fluid is recovered from this vorticity by using the Biot-Savart law. The limit of the velocity can be taken as the interface is approached, and use of the Plemelj formulas [40] then yields the Birkhoff-Rott integral. This gives the equation of motion for the interface location. For the vortex sheet strength, its evolution may also be derived from the Euler equations. Specifically, the velocity potential in each fluid region satisfies a Bernoulli equation, which is an integrated version of the Euler equations. The vortex sheet strength can be expressed as the derivative with respect to the parameter along the interface of the jump in velocity potential across the interface; the two Bernoulli equations can be subtracted to find the evolution of the jump in potential across the interface.

As we have said previously, the Duchon-Robert plan is to construct solutions on the stable manifold for the system (1), (2). This is done by making a fixed-point formulation, and using the contraction mapping theorem to demonstrate existence of this fixed point. We now develop the fixed-point equation. To start, we take the sum and difference of (1), (2):

$$(v+\omega)_t - \Lambda(v+\omega) = (F+G)_x,$$

$$(v-\omega)_t + \Lambda(v-\omega) = (F-G)_x.$$

Taking the Fourier transform of these, we have

$$\left(\widehat{v+\omega}\right)_t - |\xi|\left(\widehat{v+\omega}\right) = i\xi\left(\widehat{F+G}\right), \tag{3}$$

$$\left(\widehat{v-\omega}\right)_t + |\xi|\left(\widehat{v-\omega}\right) = i\xi\left(\widehat{F-G}\right). \tag{4}$$

We let $f = v(0,\cdot) - \omega(0,\cdot)$, and we solve (4) with an integrating factor:

$$\widehat{v-w}(t,\xi) = \hat{f}(\xi)e^{-|\xi|t} + i\xi\int_0^t e^{(s-t)|\xi|}(\widehat{F-G})(s,\xi)\ ds.$$

Taking the inverse transform, this is

$$(v-\omega)(t,\cdot) = S(t)f + (I^+(F-G))(t,\cdot). \tag{5}$$

For (3), we proceed in the same fashion but integrating backwards in time from some time $T > 0:$

$$\widehat{v+\omega}(t,\xi) = \left(\widehat{v+\omega}\right)(T,\xi)e^{(t-T)|\xi|} + i\xi\int_t^T e^{(t-s)|\xi|}\left(\widehat{F+G}\right)(s,\xi)\ ds.$$

We take the limit of this as $T \to \infty$, with the assumption that $e^{(t-T)|\xi|}(\widehat{v+w})$ goes to zero as $T \to \infty$. Also taking the inverse transform, we find

$$(v+\omega)(t,\cdot) = (I^-(F+G))(t,\cdot). \tag{6}$$

We then use (5) and (6) to solve for v and ω, finding

$$v = \frac{1}{2}Sf + \frac{1}{2}I^{+}(F-G) + \frac{1}{2}I^{-}(F+G), \tag{7}$$

$$\omega = -\frac{1}{2}Sf - \frac{1}{2}I^{+}(F-G) + \frac{1}{2}I^{-}(F+G). \tag{8}$$

We now solve for f, keeping in mind that we want to specify initial data for v, which will be $v(0,\cdot) = v_0$:

$$v_0 = \frac{1}{2}f + \frac{1}{2}I_0(F+G).$$

So, $f = 2v_0 - I_0(F+G)$. Using this with (7) and (8), we arrive at our fixed-point system,

$$v = Sv_0 - \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^{+}(F-G) + \frac{1}{2}I^{-}(F+G), \tag{9}$$

$$\omega = -Sv_0 + \frac{1}{2}SI_0(F+G) - \frac{1}{2}I^{+}(F-G) + \frac{1}{2}I^{-}(F+G). \tag{10}$$

To prove existence of solutions, we let the operator $\mathcal{T}$ be given by

$$\mathcal{T}(v,\omega) = \begin{pmatrix} Sv_0 - \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^{+}(F-G) + \frac{1}{2}I^{-}(F+G) \\ -Sv_0 + \frac{1}{2}SI_0(F+G) - \frac{1}{2}I^{+}(F-G) + \frac{1}{2}I^{-}(F+G) \end{pmatrix}.$$

We will seek fixed points of $\mathcal{T}$ by demonstrating that it is a contraction on a ball in $\mathcal{B}_\alpha$ for some $\alpha > 0$.

Of course to properly prove that $\mathcal{T}$ is a contraction, we must deal with the structure of F and G. We will not do so here, as this is highly specific to the vortex sheet problem. We will treat the requisite bound as an assumption for the present purposes:

(A1): There exists a continuous function $A : (\mathcal{B}_\alpha)^4 \to \mathbb{R}$ satisfying $A(0,0,0,0) = 0$ such that for any sufficiently small $(v_1,\omega_1) \in \mathcal{B}_\alpha$ and $(v_2,\omega_2) \in \mathcal{B}_\alpha$, we have

$$\|F(v_1,\omega_1) - F(v_2,\omega_2)\|_{\mathcal{B}_\alpha} \leq A(v_1,\omega_1,v_2,\omega_2)\left(\|v_1 - v_2\|_{\mathcal{B}_\alpha} + \|\omega_1 - \omega_2\|_{\mathcal{B}_\alpha}\right),$$

and

$$\|G(v_1,\omega_1) - G(v_2,\omega_2)\|_{\mathcal{B}_\alpha} \leq A(v_1,\omega_1,v_2,\omega_2)\left(\|v_1 - v_2\|_{\mathcal{B}_\alpha} + \|\omega_1 - \omega_2\|_{\mathcal{B}_\alpha}\right).$$

We also note that for our F and G, we have $F(0,0) = G(0,0) = 0$.

Under assumption **(A1)**, we must demonstrate two things about the mapping $\mathcal{T}$: that it maps a ball of $\mathcal{B}_\alpha \times \mathcal{B}_\alpha$ to itself, and that it has the contracting property.

We let $v_0 \in B_0$ be given, and we denote $\|v_0\|_{B_0} = r/2$. We let D equal the closed ball in $\mathcal{B}_\alpha \times \mathcal{B}_\alpha$ centered at $(Sv_0, -Sv_0)$, with radius r_1. (We will be placing conditions on r and r_1 to ensure that we have the desired properties for $\mathcal{T}$.) So the norm of the center of the ball in $\mathcal{B}_\alpha$ is equal to r, and for any point $p \in D$, we have $\|p\|_{\mathcal{B}_\alpha} \leq r + r_1$. Letting $(v,\omega) \in D$, we wish to show $\mathcal{T}(v,\omega) \in D$. It is sufficient to show the following two bounds:

$$\left\| -\frac{1}{2}SI_0(F+G) + \frac{1}{2}I^{+}(F-G) + \frac{1}{2}(F+G) \right\|_{\mathcal{B}_\alpha} \leq \frac{r_1}{2}, \tag{11}$$

$$\left\|\frac{1}{2}SI_0(F+G)-\frac{1}{2}I^+(F-G)+\frac{1}{2}I^-(F+G)\right\|_{\mathcal{B}_\alpha}\leq\frac{r_1}{2}. \tag{12}$$

Working with (11), using the operator bounds of Section 2.2, together with assumption **(A1)** and the fact that $F(0,0)=G(0,0)=0$, and using the triangle inequality, we see that for $\alpha\in(0,1)$, we have

$$\left\|\frac{1}{2}SI_0(F+G)+\frac{1}{2}I^+(F-G)+\frac{1}{2}(F+G)\right\|_{\mathcal{B}_\alpha}\leq\frac{3}{1-\alpha}A(v,\omega,0,0)(r+r_1).$$

If we take $r_1=r$ then it remains to verify

$$\frac{6}{1-\alpha}A(v,\omega,0,0)\leq\frac{1}{2}.$$

This can then be satisfied by taking r sufficiently small, since A is continuous and since $A(0,0,0,0)=0$. So, choosing $v_0\in B_0$ sufficiently small and defining the ball D in this way, we have that $\mathcal{T}$ maps D to D.

Next we examine the contracting property; we thus consider $\mathcal{T}(v_1,\omega_1)-\mathcal{T}(v_2,\omega_2)$. Again, by using the operator estimates of Section 2.2 together with the triangle inequality and assumption **(A1)**, we quickly arrive at

$$\begin{aligned}&\|\mathcal{T}(v_1,\omega_1)-\mathcal{T}(v_2,\omega_2)\|_{\mathcal{B}_\alpha\times\mathcal{B}_\alpha}\\&\qquad\leq\frac{6}{1-\alpha}A(v_1,\omega_1,v_2,\omega_2)\left(\|v_1-v_2\|_{\mathcal{B}_\alpha}+\|\omega_1-\omega_2\|_{\mathcal{B}_\alpha}\right).\end{aligned}$$

As long as

$$\frac{6}{1-\alpha}A(v_1,\omega_1,v_2,\omega_2)<1, \tag{13}$$

then $\mathcal{T}$ is contracting. Since the function A is continuous and equals zero at zero, by taking $r+r_1$ small enough, we can ensure that (13) is satisfied.

The following, then, is the Duchon-Robert theorem:

Theorem 1. *There exists $\varepsilon>0$ such that for any $v_0\in B_0$ with $\|v_0\|_{B_0}\leq\varepsilon$, there exists $(v,\omega)\in\mathcal{B}_\alpha\times\mathcal{B}_\alpha$ such that (v,ω) solve (1), (2), with $v(0,\cdot)=v_0$. Furthermore, the solution is in the ball*

$$\{(f,g)\in\mathcal{B}_\alpha\times\mathcal{B}_\alpha:\|(f,g)-(Sv_0,-Sv_0)\|_{\mathcal{B}_\alpha\times\mathcal{B}_\alpha}\leq 2\|v_0\|_{B_0}\},$$

and is unique in this ball.

As we have mentioned above, the fact that the solutions are in the space $\mathcal{B}_\alpha$ indicates two important things: that the solutions exist for all $t\in[0,\infty)$, and that for all $t\in(0,\infty)$, the solutions are analytic. Another feature of the vortex sheet problem is time-reversibility. We now combine these aspects into an ill-posedness proof in Sobolev spaces. This was not discussed in [20], but is an immediate consequence.

Theorem 2. *Let $s_1\in\mathbb{N}$, $s_2\in\mathbb{N}$ be given. The vortex sheet initial value problem does not exhibit continuous dependence on the initial data in the space $H^{s_1}\times H^{s_2}$.*

Proof. Let $v_0\in B_0$ be small and be such that v_0 is not in H^{s_1}. Let $t_*>0$ be given. The solution guaranteed by Theorem 1 is analytic at time t_*, and is still small. The solution at time t_* is therefore small in $H^{s_1}\times H^{s_2}$. Reversing time, we have a solution of the vortex sheet problem which is initially small in $H^{s_1}\times H^{s_2}$, but for which v blows up in H^{s_1} at time t_*. We see that arbitrarily small data can thus lead to arbitrarily fast blowup, demonstrating discontinuous dependence on the data. □

The Duchon-Robert result was generalized to the case in which the two fluids have different densities (still both nonzero) by Beck, Sosoe, and Wong [11]; they also extended the method to a different problem in interfacial fluid dynamics, the Muskat problem. It was also generalized in another sense, to the case of a finite time interval for the vortex sheet, in [39]; there, an exploration of which different combinations of boundary data may be specified at time $t = 0$ and at time $t = T > 0$ was made. Other authors have also considered solving the vortex sheet problem as a boundary value problem on a finite time interval for specific purposes [14], [47].

3. Boussinesq equations

There are many versions of Boussinesq equations which have been studied over the years; we will focus on a version which we may call the *abcd*-systems of Bona, Chen, and Saut [12], [13]. These systems, for real parameters a, b, c, and d, are

$$\partial_t\eta + \partial_x w + \partial_x(w\eta) + a\partial_x^3 w - b\partial_x^2\partial_t\eta = 0,$$

$$\partial_t w + \partial_x\eta + w\partial_x w + c\partial_x^3\eta - d\partial_x^2\partial_t w = 0.$$

The parameters a, b, c, and d are not completely independent, as they should satisfy $a + b + c + d = 1/3$. However we will leave all four parameters in the system for convenience. These systems are, like the vortex sheet, models for free-surface fluid dynamics, but in the water wave case, which is the case that the density of the upper fluid in Figure 1 is taken to be zero. The dependent variable η represents the location of the free surface and w represents the horizontal velocity of the fluid at a fixed depth. These are both taken to be functions of the horizontal variable, x, and time, t. The behavior of the system as regards well-posedness very much depends on the values of a, b, c, and d. Bona, Chen, and Saut in [12] developed the models and determined which parameter values give linearly well-posed initial value problems and which give linearly ill-posed initial value problems. Then in [13] they proved the nonlinear well-posedness for those systems which are linearly well-posed. Determining nonlinear ill-posedness of the remaining systems was left open at the time, and many of the cases have been proved to be nonlinearly ill-posed now by the Duchon-Robert method by the author, Bona, and Milgrom [7].

We will primarily focus on these systems not in their greatest generality, but instead choosing one particular example to treat. We set $a = 1/3$ and take $b = c = d = 0$; this system is known as the Kaup system [30]. The Kaup system is known to be completely integrable, but to be linearly ill-posed. We will show how it fits into the Duchon-Robert framework, so that a proof of nonlinear ill-posedness of the initial value problem can be achieved.

We need two modifications of the $\mathcal{B}_\alpha$ function spaces defined above; we call the new versions $\mathcal{B}_\alpha^j$. This allows for an algebraic weight in addition to the exponential weight used above; we also define these on the torus rather than on free space. For the present purpose, when we mention the torus $\mathbb{T}$, we mean the interval $[0,1]$ taken with periodic boundary conditions. Given $j \in \mathbb{N}$ we let $\mathcal{B}_\alpha^j(\mathbb{T})$ consist of all continuous-in-time functions f such that the norm

$$\|f\|_{\mathcal{B}_\alpha^j(\mathbb{T})} = \sum_{k\in\mathbb{Z}} \sup_{t\in[0,\infty)} (1 + |k|^j) e^{\alpha t|k|} |\hat{f}(k)|$$

is finite.

We write the Kaup system specifically, which is

$$\partial_t\eta + \partial_x w + \partial_x(w\eta) + \frac{1}{3}\partial_x^3 w = 0,$$

$$\partial_t w + \partial_x\eta + w\partial_x w = 0.$$

Note that the means of η and w are conserved by the evolution, and we take η and w both to have mean zero. Therefore in what follows we will take operators to act on functions of mean zero. We will now show how to fit this into the Duchon-Robert framework. One feature of the vortex sheet system was that the spatial linear operator was the same in (1) and (2); this operator was Λ. We symmetrize our system to have this same structure. To this end, we define a new variable $u = \Theta H\eta$. Here H is the Hilbert transform, which is defined in terms of its symbol as $\hat{H}(k) = -i\mathrm{sgn}(k)$. The other operator, Θ, is defined in terms of its symbol by

$$\hat{\Theta}(k) = \left(\frac{4\pi^2k^2}{3} - 1\right)^{-1/2}.$$

Note again that we are not concerned with the $k = 0$ mode, so we take this as the definition of Θ for nonzero k. Then, for all nonzero k we see that the quantity under the square root is positive.

The result of this change is the following coupled system for u and w :

$$\partial_t u - Aw = \partial_x\Theta H(wH\Theta^{-1}v), \tag{14}$$

$$\partial_t w - Au = -\frac{1}{2}\partial_x(w^2). \tag{15}$$

The operator A is the Fourier multiplier operator with symbol

$$\hat{A}(k) = (2\pi|k|)\left(\frac{4\pi^2k^2}{3} - 1\right)^{1/2}. \tag{16}$$

This system is now completely amenable to the Duchon-Robert analysis, with the caveat that we must use the torus instead of the real line to find global solutions. On the real line, or even if our periodic interval had a different length than the interval $[0, 1]$ we are considering, then some wavenumbers would lead to imaginary values of the symbol of A in (16). If we had such imaginary values, then we would need to modify our function spaces to a bounded time interval to perform our contraction proof. Ill-posedness could still be proved in such a case.

Having said that, we continue now with the Kaup system on the periodic interval $[0, 1]$, and we are able to adapt the Duchon-Robert proof to find global solutions, specifying half the data. The system (14), (15) works well with the Wiener algebra since the nonlinearities are quadratic, and thus work well with our algebra property. The operator Θ is of order -1, and the algebraic weights $|k|^j$ in our modified definition of our Wiener-type spaces helps us to deal with this. The operator A is of order two, so we expect to gain two derivatives in our Duhamel integrals. The nonlinearities only have first derivatives, so this is more than enough gain of regularity. In short, the fixed point formulation and contraction argument go through as in Section 2, yielding global solutions. We have the following theorem on the Kaup system [7]:

Theorem 3. *Let α satisfy $\alpha \in \left(0, 2\pi\sqrt{\frac{4\pi^2}{3} - 1}\right)$. There is an $r_0 = r_0(\alpha) > 0$ and a constant $C = C(\alpha)$ such that if $0 < r < r_0$, then for $w_0 \in B_0$ with $\|w_0\|_{B_0} \leq r$, there exists $(u, w) \in \mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha$ that solves the system (14), (15) with $v(0, \cdot) = v_0$ and*

$$\|(u, w)\|_{\mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha} \leq Cr. \tag{17}$$

We now state the precise ill-posedness theorem proved in [7].

Theorem 4. *The Kaup system is ill-posed in Sobolev spaces. More precisely, for any $s_1 \in \mathbb{N}$ and $s_2 \in \mathbb{N}$, there is a sequence $\{(\eta_0^n, w_0^n)\}_{n\in\mathbb{N}}$ of initial data in $H^{s_1}(\mathbb{T}) \times H^{s_2}(\mathbb{T})$ and positive times $\{t_n\}_{n\in\mathbb{N}}$, both of which tend to zero in their respective norms, such that*

$$\lim_{t \uparrow t_n} \left\|(\eta_n(\cdot, t), w_n(\cdot, t))\right\|_{H^{s_1}(\mathbb{T}) \times H^{s_2}(\mathbb{T})} = +\infty.$$

The proof is the same as the proof of Theorem 2; that is, Theorem 3 guarantees existence of small solutions which are not initially analytic but which become analytic immediately. Reversing time, we see that we may start with small smooth data and have blowup arbitrarily fast.

As we have mentioned, the nonlinear ill-posedness results of [7] encompass more of the *abcd*-systems than just the Kaup system; we present the result only for the Kaup system here for simplicity. We do mention that the method requires that the operator A, which we may define for any of the *abcd*-systems by symmetrizing, must be of positive order for the Duchon-Robert method to work. There are some linearly ill-posed *abcd*-systems for which the associated A is of order zero; in such cases, we expect the systems are nonlinearly ill-posed but the present method is not effective.

Finally, we mention that a related work in the literature on Boussinesq equations is the center manifold theorem by de la Llave [19].

4. Parabolic problems

We have described so far how the method of Duchon and Robert applies to problems which are elliptic in space-time, but it applies just as well to parabolic problems. Consider equations of the form

$$u_t = Au + N(u), \tag{18}$$

where A is a negative elliptic operator (such as the Laplacian) and N is a nonlinear operator. Take this with initial data $u(0, \cdot) = u_0$. Then we can write the Duhamel formula

$$u(t, \cdot) = e^{At}u_0 + \int_0^t e^{(t-s)A} N(u(s, \cdot))\, ds. \tag{19}$$

This is perhaps a more classical version of a Duhamel formula, in which we only need to integrate forward in time from the initial time ($t = 0$), as compared to the forward-backward Duhamel formula (9), (10) developed for the vortex sheet.

The Duhamel formula (19) is a fixed-point formulation of the forward parabolic equation (18), and we can use the Duchon-Robert method to prove existence of solutions. To do so, we need some assumptions on the nonlinearity N. First, the number of derivatives present in N must be no more than the number of derivatives present in A; as seen above for the estimates of I^+ in Section 2, when we integrate the symbol of the semigroup in time, we gain as many derivatives as are present in A. So, if N has this many derivatives or fewer, then we may conclude that the Duhamel integral is a bounded operator. Also, to prove that we have a contraction mapping, we should be able to bound a norm of $N(u)-N(v)$ by a norm of $u-v$ times a small constant. If N is quadratic, say, then this constant would be a norm of $u+v$, for instance, and we could then have a contraction if u and v are sufficiently small, i.e. we would be able to prove the map is contracting in a small ball.

We now give some examples of parabolic problems which fit into this framework. In addition to these examples, the author and Mazzucato have treated the two-dimensional Kuramoto-Sivashinsky equation with the Duchon-Robert framework [8].

4.1. **A problem from epitaxial growth.** The following equation has been put forward as a model for epitaxial growth of a thin film [31], [38]:

$$h_t = \Delta e^{-\Delta h}. \tag{20}$$

Here, h represents the height of the thin film at each location in space at each time. Of course, the equation is taken with initial data

$$h(0,\cdot) = h_0. \tag{21}$$

Expanding the exponential with its Taylor series, (20) becomes

$$h_t = -\Delta^2 h + \sum_{k=2}^{\infty} \Delta \left(\frac{(-\Delta h)^k}{k!} \right). \tag{22}$$

We have applied the Duchon-Robert method to the problem in this form [4], using the function spaces $\mathcal{B}^j_\alpha$ on the torus as in Section 3. We will take this opportunity to verify the analogue of the assumption **(A1)** from Section 2 for this problem. We need to estimate a difference of two copies of the sum in (22), but we may neglect the Laplacian (because of the gain of derivatives in the Duhamel integral, just as we estimated a derivative contained in I^+ in Section 2 above). We wish to establish a bound

$$\left\| \sum_{k=2}^{\infty} \left(\frac{(-\Delta h_1)^k - (-\Delta h_2)^k}{k!} \right) \right\|_{\mathcal{B}^0_\alpha} \leq A(h_1,h_2)\|h_1-h_2\|_{\mathcal{B}^2_\alpha}, \tag{23}$$

where $A(h_1,h_2)$ goes to zero as h_1 and h_2 go to zero.

We will use the formula

$$a^k - b^k = (a-b)\sum_{j=0}^{k-1} a^{k-1-j} b^j \tag{24}$$

in bounding (23). Using the triangle inequality, (24), and the fact that $\mathcal{B}^0_\alpha$ is an algebra, we have the bound

$$\begin{aligned}\left\| \sum_{k=2}^{\infty} \left(\frac{(-\Delta h_1)^k - (\Delta h_2)^k}{k!} \right) \right\|_{\mathcal{B}^0_\alpha} &\leq \sum_{k=2}^{\infty} \frac{1}{k!} \left\| (\Delta h_1)^k - (\Delta h_2)^k \right\|_{\mathcal{B}^0_\alpha} \\ &\leq \|h_1-h_2\|_{\mathcal{B}^2_\alpha} \sum_{k=2}^{\infty}\sum_{j=0}^{k-1} \|h_1\|_{\mathcal{B}^2_\alpha}^{k-1-j} \|h_2\|_{\mathcal{B}^2_\alpha}^{j}.\end{aligned}$$

We let $R = \max\left\{\|h_1\|_{\mathcal{B}^2_\alpha}, \|h_2\|_{\mathcal{B}^2_\alpha}\right\}$, and we continue to estimate:

$$\left\|\sum_{k=2}^{\infty}\left(\frac{(-\Delta h_1)^k - (\Delta h_2)^k}{k!}\right)\right\|_{\mathcal{B}^0_\alpha} \leq \|h_1 - h_2\|_{\mathcal{B}^2_\alpha}\sum_{k=2}^{\infty}\frac{R^{k-1}}{(k-1)!} = (e^R - 1)\|h_1 - h_2\|_{\mathcal{B}^2_\alpha}.$$

We have thus achieved our goal of arriving at (23), with the function A given by

$$A(h_1, h_2) = \exp\left\{\max\left\{\|h_1\|_{\mathcal{B}^2_\alpha}, \|h_2\|_{\mathcal{B}^2_\alpha}\right\}\right\} - 1.$$

With this estimate, we are able to apply the Duchon-Robert framework to arrive at an existence theorem guaranteeing that sufficiently small data leads to solutions for all time. By tracking constants in the estimates carefully, we can even be explicit about the smallness condition. We have the main theorem of [4]:

Theorem 5. *Let h_0 satisfy $\|\Delta h_0\|_{B_0(\mathbb{T}^n)} < 1/4$. Let $\alpha \in (0,1)$ be such that*

$$\|\Delta h_0\|_{B_0(\mathbb{T}^n)} = \frac{1-\alpha}{2(2-\alpha)}.$$

Then there exists $h \in \mathcal{B}^2_\alpha$ such that h solves (20) with initial data (21). For any $t > 0$, the function $h(t, \cdot)$ is analytic, with radius of analyticity greater than or equal to αt.

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Other authors have studied this model as well. Liu and Strain proved existence of small global solutions on free space with Δh_0 in the Wiener algebra, with the norm of Δh_0 at most $1/10$, and also proved analyticity of solutions [36]. Granero-Belinchón and Magliocca proved existence of small global solutions on the torus, with Δh_0 of size at most $1/10$ in the Wiener algebra [26].

4.2. **Mean field games.** Mean field games are problems from game theory in which the interaction of a large number of similar agents is approximated by viewing one representative agent as interacting with the bulk of the other agents; this is in analogy with mean field systems arising in physics. Mean field games were introduced independently by Lasry and Lions in [32], [33], [34] and by Caines, Huang, and Malhamé in [27], [28].

The unknowns in the mean field games system are the distribution of all the agents, m, and the value function, u, that a representative agent is optimizing. The mean field games PDE system consists of a Hamilton-Jacobi equation for u and a Fokker-Planck equation for m :

$$\partial_t u + \Delta u + H(t, x, m, \nabla u) = 0, \tag{25}$$

$$\partial_t m - \Delta m + \operatorname{div}(m H_p(t, x, m, \nabla u)) = 0. \tag{26}$$

Here the nonlinear function H is known as the Hamiltonian, and H_p denotes the gradient with respect to the p variables when we write $H = H(t, x, m, p)$. For simplicity we take the spatial domain to be the d-dimensional torus, $\mathbb{T}^d$, and given $T > 0$ we take the temporal domain to be $[0, T]$. Notice that (26) is a forward parabolic equation for m while (25) is a backward parabolic equation for u. Therefore appropriate data for the system is initial data for the distribution m and terminal data for the value function u,

$$m(0, \cdot) = m_0, \qquad u(T, \cdot) = G(m(T, \cdot), \cdot), \tag{27}$$

where G is known as the payoff function. A simple case is when G does not actually depend on $m(T,\cdot)$, and thus $u(T,\cdot) = u_T$ for a given function u_T.

A number of existence theorems for the system (25), (26) assume structure on the Hamiltonian, H, namely that it is additively separable into a part which depends on ∇u and a part which depends on m; the part depending on ∇u is then typically taken to be convex while the part depending on m is taken to be monotone. Existence of solutions can then be proven using methods relying on convexity and monotonicity; a few examples are [17], [21], [22], [23], [41], [42], [43].

As an alternative to assuming structure on the Hamiltonian, we may prove existence of solutions for mean field games by placing smallness constraints as in the Duchon-Robert method. Notice that the mean field games system combines aspects of the elliptic problems we have studied above and the forward parabolic problems as well. In particular, the equations are parabolic, but we need to consider them in a forward-backward way. This system is therefore amenable to a suitable adaptation of the Duchon-Robert method [2], [5]. We mention that there are some other proofs of existence of solutions for the nonseparable mean field games system in the literature by methods other than using the Duchon-Robert method, especially [1], [3], [6], [18], [24], [25].

To adapt the Duchon-Robert framework for mean field games, we again need a modification of the above function spaces. This modification is necessary because we are solving a forward-backward problem over a finite time interval, and we wish to have non-analytic data specified at both ends of this interval while still having analytic solutions in the interior of the interval. We will define $\mathcal{B}^j_{\alpha,T}(\mathbb{T}^d)$, but first we introduce an auxiliary function, β. Given $\alpha > 0$, we define $\beta : [0,T] \to [0,\alpha]$ by

$$\beta(s) = \begin{cases} 2\alpha s/T, & s \in [0, T/2], \\ 2\alpha - 2\alpha s/T, & s \in [T/2, T]. \end{cases}$$

(The graph of β is shown in Figure 2.) Then $\mathcal{B}^j_{\alpha,T}$ is the set of all continuous-in-time functions f for which the norm

$$\|f\|_{\mathcal{B}^j_{\alpha,T}(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d} \sup_{t\in[0,T]} (1+|k|^j)e^{\beta(t)|k|}|\hat{f}(t,k)|$$

is finite. So, for a function $f \in \mathcal{B}^j_{\alpha,T}(\mathbb{T}^d)$, it need not be analytic at time $t = 0$ or at time $t = T$, but is analytic with radius of analyticity at least $\beta(t)$ for all $t \in (0,T)$. This lower bound on the radius of analyticity grows linearly from both ends of the interval, peaking at the midpoint, with a maximal guaranteed rate of decay of the Fourier series at time $t = T/2$.

In these spaces, we can use the Duchon-Robert method to prove the main theorem of [5]:

Theorem 6 (Informal statement). *Let $T > 0$ and $\alpha \in (0, T/2)$ be given. Assume that the Hamiltonian, H, and the payoff function, G, satisfy appropriate Lipschitz properties. Let $\bar{m}$ denote the uniform distribution on $\mathbb{T}^d$. There exists $\delta > 0$ such that if m_0 is a probability measure such that $\mu_0 = m_0 - \bar{m}$ satisfies $\|\Delta\mu_0\|_{B_0} < \delta$, then the system (25), (26), (27) has a strong, locally unique solution $(u,m) \in \mathcal{B}^2_\alpha \times \mathcal{B}^2_\alpha$. Furthermore, for all $t \in (0,T)$, each of $u(t,\cdot)$ and $m(t,\cdot)$ are analytic, and $m(t,\cdot)$ is a probability measure.*

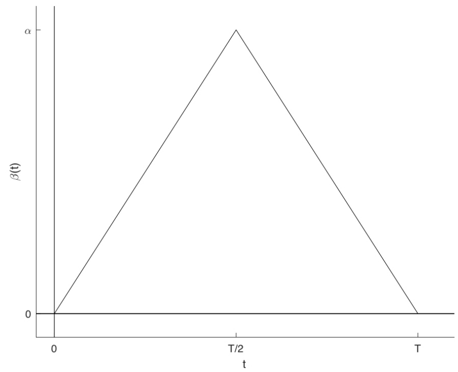

FIGURE 2. The graph of $\beta(t)$.

5. Conclusion

The framework developed by Duchon and Robert in [20] is useful for proving existence theorems for nonlinear problems which are either parabolic or elliptic on space-time. We have demonstrated applications of this in the original vortex sheet setting considered by Duchon and Robert as well as for families of Boussinesq equations, a problem in epitaxial growth, and mean field games. The vortex sheet and the Boussinesq equations, which are elliptic in space-time, have the added feature that the existence theory via the Duchon-Robert method also immediately implies ill-posedness of the associated initial value problems in Sobolev spaces.

This method uses function spaces based on the Wiener algebra. Being an algebra, it is well-suited for nonlinearities which are polynomial, as in the Boussinesq problems described above, or if the nonlinearities can be expanded as a power series. We have shown the use of such an expansion in the epitaxial growth problem of Section 4.1, but this was also how Duchon and Robert used the method; in Section 2 we omitted details of the treatement of the specific nonlinearity for the vortex sheet, but this does come down to making series expansions for the relevant singular integrals.

The essential steps of the Duchon-Robert method are to make a fixed-point formulation by a Duhamel formula, to prove that the Duhamel integral is bounded on the Wiener-type spaces, and to prove a Lipschitz property in the Wiener-type spaces for the nonlinearity. To demonstrate the boundedness of the Duhamel integral, an essential ingredient is that the spatial linear operator (which was Λ in the vortex sheet problem, $-\Delta^2$ in the epitaxial growth problem, and Δ in the mean field games problem, for example) has at least as many derivatives as are present in the nonlinear terms.

Many problems arising in practice have the properties we have just mentioned, so the Duchon-Robert framework can be used as a relatively efficient method for demonstrating existence theory.

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