NONEXISTENCE OF SMALL, SMOOTH, TIME-PERIODIC, SPATIALLY PERIODIC SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We study the question of nonexistence of small spatially periodic, time-periodic solutions for cubic nonlinear Schrödinger equations. We prove that for almost any value in a bounded set of possible temporal periods, there is an amplitude threshold, below which any initial value is not the initial value for a time-periodic solution. The proof requires a certain level of Sobolev regularity on solutions. The methods used are not based on any special structure of the nonlinear Schrödinger equation, and can be applied more generally.

1. INTRODUCTION

We study time-periodic solutions of the cubic nonlinear Schrödinger equation,

$$u_t = -i\Delta u + \beta i \left|u\right|^2 u, \quad \text{where} \quad \mathbf{x} \in \mathbb{T}^n,\ t \in \mathbb{R}. \tag{1}$$

Depending on the choice of the constant, β, this either is the focusing or the defocusing case; this distinction makes no difference for the analysis we perform, so we leave β to satisfy $\beta \in \{-1, 1\}$.

In prior work, the authors introduced a framework for demonstrating non-existence of small-amplitude, time-periodic solutions for nonlinear dispersive equations, for almost every temporal period [2], [3]. This framework was developed with the Korteweg-de Vries (KdV) equation in mind, and was applied to the KdV equation as well as some similar equations such as the Kawahara equation. This framework used in a fundamental way that the KdV evolution preserves the mean of solutions; for the related equations we studied, we maintained this feature. We mention that for some nonlinear wave equations, de la Llave has shown the nonexistence of small, time-periodic solutions for almost every temporal period [9]; this work used a variational method which appears to be completely different from the current method.

We now wish to extend the results of [2], [3] to the case of Schrödinger equations. Since nonlinear Schrödinger evolution does not preserve the mean of solutions, we must make additional efforts as compared to the proofs of [2], [3]. We therefore follow the prior framework as much as possible to a point, and then make a departure to follow considerations more relevant for Schrödinger equations.

The framework of [2], [3] started from the idea that time-periodic solutions are fixed points of the Poincare map, and thus we rewrote the problem of finding time-periodic solutions as a fixed point problem. The operator in question included a linear factor and a

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nonlinear factor, and these were estimated in different ways to demonstrate a contracting property at small amplitudes. Since the operator is a contraction, and since the trivial (zero) solution is time-periodic, there could be no other small-amplitude time-periodic solutions with the given temporal period.

The linear factor was estimated with a small divisor technique, leading to a loss of regularity. This regularity was recovered with a dispersive smoothing estimate for the linear part. The dispersive smoothing estimate modified an argument of Erdogan and Tzirakis that the Duhamel integral for KdV evolution on the torus gains almost one derivative [10]. Notice that the method of the present work and of [2], [3] hinges only on whether certain estimates, such as those just discussed for the Duhamel integral, can be established, and we rely on no special structure such as integrability or near-integrability. While we treat only (1) in the present contribution, the techniques developed apply to more general equations, including other Schrödinger equations, such as with different nonlinearities or with higher-order linear parts.

We mention that (1) possesses a family of spatially-independent time-periodic solutions. For $B \in \mathbb{C}$, it can be checked that $u(x,t) = B\exp\{|B|^2\beta it\}$ solves (1), and is clearly time-periodic. The temporal period is $T = \dfrac{2\pi}{\beta|B|^2}$, which can take any non-negative value, depending on the choice of B. Fortunately, the existence of these solutions is not a counterexample to the theorem we develop – for these solutions, as the amplitude goes to zero, the temporal period then goes to positive infinity. In our main theorem, we rule out the existence of arbitrarily small nontrivial time-periodic solutions for almost every period, with the possible temporal periods chosen from a bounded set. We mention also that we have some unformity in the smallness constraint, given that the temporal period comes from a bounded set. Thus, our theorem can be viewed as a rigorous explanation of the fact that as the amplitude of time-periodic solutions go to zero, the temporal periods must go to infinity, as happens for this explicit family.

The proof requires a certain level of Sobolev regularity on solutions, so it remains theoretically possible that small, rough, time-periodic, spatially periodic solutions could exist for these temporal periods. It is frequently the case that coherent waves (such as traveling waves or time-periodic waves) for nonlinear dispersive equations are analytic, with exceptions such as compactons for Rosenau-Hyman equations [14], [15] or peakons for the Camassa-Holm equation [7] seen as being related to the presence of degeneracy in the evolution equations [12]. Our regularity requirement can be seen as stemming from two sources. One of these is the fact that our estimate for the inverse of a certain linear operator costs derivatives, so these derivatives must be compensated for in our estimate of the Duhamel integral. However, to fully compensate for this derivative loss appears to require fairly strong dispersion (for instance, we treat an equation with fifth-order dispersion in [2]). In the case of second-order dispersion as in Schrödinger equations, our estimate for the Duhamel integral is much cruder, and actually costs further derivatives.

One motivation for the present study is that the question of existence or nonexistence of time-periodic solutions for dispersive equations has a number of unanswered questions at present, and we seek to fill in some additional details of the picture. In particular, for a number of equations, time-periodic solutions have primarily been shown to exist by Nash-Moser-type methods [4], [5], [6], [8], [17]. Nash-Moser type methods give existence of solutions for a set of parameters of positive measure (perhaps full measure), but do not speak to the existence or non-existence of solutions for remaining parameter values. Our method is complementary, showing non-existence of solutions for most periods at small amplitudes, and thus providing additional information about where solutions exist or not in amplitude-frequence space.

Another motivation is the difficulty in understanding dynamics of nonlinear dispersive equations on the torus. (We refer the interested reader to [13] for a recent survey of the study of nonlinear Schrödinger equations on the torus.) On free space, quite a large number of scattering results have been proven; we mention just [16] as an example. There, and in similar papers, small solutions of certain dispersive equations are shown to decay to zero as time goes to infinity. We make two points about this: first, one does not have an expectation that this would happen on the torus, and second, this decay of solutions implies the nonexistence of small coherent structures. Thus, while we cannot study the decay of solutions of nonlinear Schrödinger equations on the torus, we can study the nonexistence of small coherent structures on the torus.

1.1. **Preliminaries.** Let $n \in \mathbb{N}$, with $n \geq 1$. We consider functions $f : \mathbb{T}^n \to \mathbb{C}$ where $\mathbb{T}$ is the 2π-periodic interval. We will make use of the Fourier series representation for such functions:

$$f(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{ik\cdot x}, \quad \text{where} \quad \hat{f}(k) := \frac{1}{(2\pi)^n} \int_0^{2\pi} f(x) e^{-ik\cdot x} dx.$$

We will use the notation $\langle\!\langle f \rangle\!\rangle$ for the mean of a periodic function, f :

$$\langle\!\langle f \rangle\!\rangle = \frac{1}{\mathrm{vol}(\mathbb{T}^n)} \int_{\mathbb{T}^n} f(x)\ dx.$$

We let $\mathbb{P}$ be the projection which removes the mean of a periodic function:

$$\mathbb{P} f = f - \langle\!\langle f \rangle\!\rangle.$$

We will work with L^2-based Sobolev spaces, H^k, with $k \in \mathbb{N}$. We denote the H^k norm as $\|\cdot\|_k$, and we also will use the L^∞ norm, which we denote $|\cdot|_\infty$. We will use the particular space H^s many times in the sequel, where s is defined as

$$s = \left\lfloor \frac{n}{2} \right\rfloor + 1. \tag{2}$$

This choice allows us to use Sobolev embedding, so that for all $f \in H^s$,

$$|f|_\infty \leq c_S \|f\|_s, \tag{3}$$

where, in case there is any doubt, we take the constant c_S to satisfy $c_S \geq 1$. For $m > 0$, we denote the norm on H^m as $\|\cdot\|_m$, defined for $f \in H^m$ through the equation

$$\|f\|_m^2 = \|f\|_{L^2}^2 + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{2m} |\hat{f}(k)|^2. \tag{4}$$

We will use some interpolation results for Sobolev spaces, and the following elementary interpolation result is useful. The proof of Lemma 1 can be found in many places, and one such place is [1].

Lemma 1 (Elementary interpolation lemma)**.** *Let $0 < \alpha < \beta$ be given. There exists $c > 0$ such that for all $f \in H^{\beta}$, the following estimate is satisfied:*

$$\|f\|_{\alpha} \leq c\|f\|_0^{1-\alpha/\beta}\|f\|_{\beta}^{\alpha/\beta}.$$

The elementary interpolation lemma allows us to prove the following interpolation result, which is the version of interpolation which will be used in the sequel.

Corollary 2 (Interpolation)**.** *Let $\sigma \geq 0$ be given. There exists $c > 0$ such that for all $g \in H^{\sigma+2n+2}$, the following estimate is satisfied:*

$$\|g\|_{\sigma+n+1} \leq c\|g\|_{\sigma}^{1/2}\|g\|_{\sigma+2n+2}^{1/2}.$$

Proof. Let $g \in H^{\sigma+2n+2}$ be given. We use Lemma 1 with $f = (-\Delta)^{\sigma/2}g$, $\alpha = n+1$, and $\beta = 2n+2$. This gives the following bound:

$$\|(-\Delta)^{\sigma/2}g\|_{n+1} \leq c\|(-\Delta)^{\sigma/2}g\|_0^{1/2}\|(-\Delta)^{\sigma/2}g\|_{2n+2}^{1/2}.$$

The corollary then follows from elementary considerations. □

We also will use an elementary lemma about balls in the complex plane.

Lemma 3. *Let $z \in \mathbb{C}$ and $r > 0$ be given, such that $z \neq 0$ and $r < \frac{|z|}{\sqrt{2}}$. Then, the closed ball $B(z;r)$ cannot intersect both the real axis and the imaginary axis.*

Proof. Let $z = a + ib$. The closest point to z on the real axis is a, and the closest point on the complex axis to z is bi. If a is in the ball, then using the definition of a ball in the complex plane and the assumed bound on r, we see that $b^2 < \frac{a^2+b^2}{2}$. If bi is in the ball, then we similarly conclude that $a^2 < \frac{a^2+b^2}{2}$. If these are both true, then adding the expressions, we see that $a^2 + b^2 < a^2 + b^2$. This proves the claim. □

2. General considerations

In this section we present the general framework for a nonexistence theorem for small spatially periodic, time-periodic solutions of an evolution equation. This is adapted from the authors' previous works [2], [3]; in those papers, KdV-type equations were treated, and an essential feature of KdV-type evolutions that was used was conservation of the mean of the solutions. In the present case, we do not assume that the mean of the solution is conserved under the evolution, since we will apply this general framework to cubic nonlinear Schrödinger equations.

2.1. **The formulation of the problem.**

We consider the generic evolution

$$v_t = Av + Nv, \tag{5}$$

where A is a linear operator and N is nonlinear. As we have noted above, we do not assume that the evolution (5) maintains the mean of solutions; however, we do assume that the linear evolution maintains the mean of solutions. That is, we assume the following:

Assumption 1. *If w is a solution of*

$$w_t = Aw,$$

then

$$\langle\langle w(\cdot,t)\rangle\rangle = \langle\langle w(\cdot,0)\rangle\rangle, \qquad \forall t. \tag{6}$$

We also make another, related assumption on the linear part, and that is simply that the semigroup e^{At} commutes with the projection $\mathbb{P}$.

Assumption 2. *For any $t > 0$, the operators e^{At} and $\mathbb{P}$ commute.*

We remark that for Schrödinger equations, we have $A = i\Delta$, and it is immediate that in this case both Assumption 1 and Assumption 2 are satisfied.

For a solution, v, of (5), with initial condition v_0, we can write the usual Duhamel formula:

$$v(\cdot,t) = e^{At}v_0 + \int_0^t e^{A(t-\tau)}N(v(\cdot,\tau))\, d\tau.$$

We rewrite this symbolically as

$$v(\cdot,t) = S_L(t)v_0 + S_D(t)v_0; \tag{7}$$

here, S_L is the semigroup operator for the linear part, $S_L(t) = e^{At}$, and $S_D(t)$ is the Duhamel integral,

$$S_D(t)v_0 = \int_0^t e^{A(t-\tau)}N(v(\cdot,\tau))\, d\tau. \tag{8}$$

Note that although the right-hand side of (8) does not explicitly depend on v_0, as long as the initial value problem for (5) is well-posed, the right-hand side is in fact completely determined by t and v_0. The time-periodic ansatz for a solution of (5) is the statement $v(\cdot,T) = v(\cdot,0)$, for some $T > 0$. Using this ansatz with (7), we have

$$v_0 = S_L(T)v_0 + S_D(T)v_0.$$

We rearrange this simply by placing everything on the left-hand side:

$$v_0 - S_L(T)v_0 - S_D(T)v_0 = 0. \tag{9}$$

Note that (6) implies $\langle\langle v_0\rangle\rangle = \langle\langle S_L(T)v_0\rangle\rangle$. Using this in (9), we conclude that $\langle\langle S_D(T)v_0\rangle\rangle = 0$, when v_0 and T determine a time-periodic solution. Applying the operator $\mathbb{P}$ to (9), we have

$$(I - S_L(T))\mathbb{P}v_0 - S_D(T)v_0 = 0. \tag{10}$$

Note that we have used Assumption 2 here.

Next, we would like to factor out the operator $I - S_L(T)$ from (10). Naturally, we can only do this if the operator $I - S_L(T)$ happens to be invertible. We define a set W as

$$W = \{T \in (0,\infty) : \ker(I - S_L(T)) = \{0\}\}.$$

Then, for $T \in W$, we can rewrite (10) as follows:

$$(I - S_L(T))\left(\mathbb{P}v_0 - (I - S_L(T))^{-1}S_D(T)v_0\right) = 0. \tag{11}$$

Based on this, we define a nonlinear operator $K(T)$:

$$K(T) = (I - S_L(T))^{-1}S_D(T).$$

Thus, if $T \in W$ and v_0 is the initial data for a time-periodic solution of (5) with temporal period T, we have

$$\mathbb{P}v_0 = K(T)v_0,$$

from which it trivially follows that

$$\|\mathbb{P}v_0\|_X = \|K(T)v_0\|_X, \tag{12}$$

for an appropriately chosen Banach space, X.

Remark 1. *In the previous papers* [2], [3]*, we considered solutions of KdV-type equations which had zero mean. Thus, in the equation corresponding to* (12)*, the projection* $\mathbb{P}$ *was absent. We showed that the operator* K *had a local contracting property, and thus in a ball around the origin, the only time-periodic solution with period* $T \in W$ *in such a ball is zero. In the present case, we need additional steps to establish this result.*

2.2. **Abstract result.** We state now the estimates which must be satisfied by both the linear part of the evolution and the nonlinear part of the evolution.

Assumption 3 (Linear Estimate)**.** *Let* $T \in W$ *be given. There exists* $p > 0$ *and* $d_1 > 0$ *such that for all* $k \in \mathbb{Z}^n \setminus \{0\}$,

$$\left|\mathcal{F}\left(I - S_L(T)\right)^{-1}(k)\right| \le d_1|k|^p. \tag{13}$$

Clearly, this linear estimates states that the operator $(I - S_L(T))^{-1}$ maps from a Sobolev space H^ω to $H^{\omega-p}$, for the given p. In [3], we have proved that Assumption 3 holds when the operator A is defined by

$$\mathcal{F}(A)(k) = i|k|^\zeta \tag{14}$$

for all $k \in \mathbb{Z}^n$, with $\zeta \in (0,\infty)$. This is the content of Lemma 1 from [3], which we now state. This is a version of classical small-divisor estimates [11].

Lemma 4. *Let the linear operator* A *be given by (14). Let* $0 < T_1 < T_2$ *be given. Let* $0 < \delta < T_2 - T_1$ *be given. Let* $p > n$ *be given. There exists a set* $W_{p,\delta} \subseteq [T_1, T_2] \cap W$ *and there exists* $c_1 > 0$ *such that the Lebesgue measure of* $W_{p,\delta}$ *satisfies* $\mu(W_{p,\delta}) > T_2 - T_1 - \delta$*, and for all* $k \in \mathbb{Z}^n \setminus \{0\}$*, for all* $T \in W_{p,\delta}$*, we have*

$$\left|\mathcal{F}(I - S_L(T))^{-1}(k)\right| < c_1|k|^p.$$

In the definition of $K(T)$ above, the linear operator is composed with $S_D(T)$; thus, to balance a loss of derivatives from the linear part, we compensate with a weak kind of gain of derivatives from the nonlinear operator. Specifically, we need to know that $S_D(T)$ can be estimated by a product of terms, one of which gains p derivatives.

Assumption 4 (Nonlinear Estimate)**.** *Let* $T \in W$ *be given. Let* p *be as in Assumption 3. There exists* $\tilde{p} \ge 0$, $q > 0$, $d_2 > 0$, *and* $\eta > 0$ *such that for all* $v_0 \in H^{s+\tilde{p}}$ *which satisfy* $\|v_0\|_{H^{s+\tilde{p}}} \le \eta$*, the following estimate is satisfied:*

$$\|S_D(T)v_0\|_{s+p} \le d_2\|v_0\|_s\|v_0\|_{s+\tilde{p}}^q.$$

Remark 2. *We make a comment to help the reader keep the parameters* p, $\tilde{p}$, *and* q *straight. We take initial data in the space* $H^{s+\tilde{p}}$*, where* s *is defined by* (2)*. As mentioned previously, we allow a loss of regularity, so that the Duhamel integral maps from* $H^{s+\tilde{p}}$ *to*

H^{s+p}, with the estimate of Assumption 4, with the expectation that $p < \tilde{p}$. The constant q is simply a measure of the nonlinearity, so that a cubic nonlinearity would correspond to $q = 2$, a quartic nonlinearity to $q = 3$, and so on.

We are now able to state and prove our abstract result.

Lemma 5 (Abstract Lemma). *Let $T_* > 0$ be given. Let $T \in W \cap [0, T_*]$ be given. Assume that each of Assumption 1, Assumption 2, Assumption 3, and Assumption 4 hold. Let d_1, d_2, η, p, and $\tilde{p}$ be as in Assumptions 3 and 4. Assume that there exists $r_1 > 0$ such that if $\|u_0\|_{s+\tilde{p}} < r_1$, then there exists a unique $u \in C([0, T_*]; H^{s+\tilde{p}})$ which solves (5) with initial data u_0.*

Then, if v_0 is the initial data for a time-periodic solution of (5), with temporal period T, satisfying $\|v_0\|_{s+\tilde{p}} < \min\{\eta, r_1\}$, then

$$\|v_0 - \langle\!\langle v_0 \rangle\!\rangle\|_s \leq d_1 d_2 \|v_0\|_s \|v_0\|_{s+\tilde{p}}^q.$$

Proof. From Assumption 3, we see that $(I - S_L(T))^{-1}$ is a bounded linear operator from H^{s+p} to H^s. Considering (12) and the definition of the operator $K(T)$, this implies that $\mathbb{P}v_0$ is bounded in terms of $S_D(T)$:

$$\|\mathbb{P}v_0\|_s \leq d_1 \|S_D(T)v_0\|_{s+p}. \tag{15}$$

The lemma now follows by applying Assumption 4, and by using the identity $\mathbb{P}f = f - \langle\!\langle f \rangle\!\rangle$. □

Remark 3. *If we considered solutions with mean zero, then the result of Lemma 5 would guarantee that initial data v_0 corresponding to nontrivial time-periodic solutions of (5) with temporal period T may not be arbitrarily small. That is, for such $v_0 \neq 0$ with $\langle\!\langle v_0 \rangle\!\rangle = 0$, if $d_1 d_2 \|v_0\|_{s+\tilde{p}}^q < 1$, then we have found a contradiction, $\|v_0\|_s < \|v_0\|_s$. Thus, we must have $\|v_0\|_{s+\tilde{p}}^q \geq 1/(d_1 d_2)$ for such v_0. This is a simplified version of the argument from [2], [3], in which the authors proved that KdV-like equations do not have arbitrarily small time-periodic solutions for almost every temporal period. Since Schrödinger equations do not maintain the mean of solutions, we still use the result of Lemma 5, but there is more to do in this case.*

3. The main theorem and its proof

Recall the definition of s in (2), and recall that we have used μ to denote Lebesgue measure on $\mathbb{R}^n$. We now state our main theorem.

Theorem 6 (Main Theorem). *Let $[T_1, T_2] \subset (0, \infty)$ be given. Let W be as above. Let $\delta > 0$ be given. There exists $W_\delta \subset W \cap [T_1, T_2]$ such that $\mu(W_\delta) > T_2 - T_1 - \delta$, and such that there exists $r_2 > 0$ such that for all $T \in W_\delta$, for all $v_0 \in H^{s+2n+2}$ which satisfy $\|v_0\|_{s+2n+2} < r_2$, if v_0 is the initial data for a time-periodic solution of (1) with temporal period T, then $v_0 = 0$.*

Note that the constant r_2 is uniform over W_δ, but can depend on T_2 and δ. By taking a union as T_2 becomes larger and δ becomes smaller, one arrives at a non-uniform version of the theorem: for almost every possible temporal period $T > 0$, there is an amplitude threshold below which there are no nontrivial time-periodic solutions for that period, and this threshold depends upon the period. The remainder of the section is devoted to the proof of Theorem 6.

3.1. **The linear operator.** For nonlinear Schrödinger equations, the operator A is proportional to $i\Delta$. In this case, it is immediate that Assumption 1 and Assumption 2 are satisfied. That Assumption 3 holds is a consequence of Lemma 4; as we have remarked before, this is related to classical small-divisor estimates [11]. In using Lemma 4, we let $p = n+1$, and we thus find the existence of a set $W_{n+1,\delta}$. We then define $W_\delta = W_{n+1,\delta}$. Lemma 4 also guarantees the existence of a constant $d_1 > 0$ for which (13) holds, for all $T \in W_\delta$.

3.2. **The doubling estimate.** Since the initial value problem for (1) is well-known to be well-posed in a variety of function spaces, we will not prove the well-posedness in the spaces of current interest, namely H^s or H^{s+2n+2}. We do, however, need to remark about the time it takes for a small solution to double in size.

We define an energy functional which is equivalent to the square of the H^{s+2n+2} norm:

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{T}^n} |v|^2 + \left|(-\Delta v)^{\frac{s}{2}+n+1} v\right|^2 \ dx.$$

Usual considerations then imply a bound for the growth of the energy:

$$\frac{d\mathcal{E}}{dt} \le c\mathcal{E}^2. \tag{16}$$

Some of the implications of (16) are that solutions cannot blow up immediately, and that solutions initially of size ε exist at least on a time interval of size proportional to $\frac{1}{\varepsilon}$. Another specific consequence, which is discussed in more detail in [3] in the context of a fifth-order dispersive equation, is that small solutions grow slowly, so that the time a solution takes to double in size goes to infinity as the initial data goes to zero. In particular, there exists $r_* > 0$ such that if $\|v_0\|_{s+2n+2} < r_*$, then the solution v exists in the space $C([0,T_2]; H^{s+2n+2})$, with the estimate

$$\sup_{t\in[0,T_2]} \|v(\cdot,t)\|_{s+2n+2} \le 2\|v_0\|_{s+2n+2}.$$

The energy analysis can be repeated with the square of the H^s norm, finding that there exists $r_{**} > 0$ such that if $\|v_0\|_s < r_{**}$, then the solution v exists in the space $C([0,T]; H^s)$, with the estimate

$$\sup_{t\in[0,T_2]} \|v(\cdot,t)\|_s \le 2\|v_0\|_s.$$

Then, we choose r_1 and η to be $r_1 = \eta = \min\{r_*, r_{**}\}$.

3.3. **Estimate for the Duhamel integral.** In [3], we made a careful study of the Duhamel integral, following [10] in making a normal form transformation and inferring gain of regularity. An advantage of that approach is that the regularity requirements on solutions of KdV-like equations in [3] were significantly lower than the regularity requirements on solutions of (1) in the present work. However, it is not clear that sufficiently strong smoothing properties hold for the Duhamel integral for nonlinear Schrödinger equations in n spatial dimensions. Therefore, rather than use the normal form transformation, we write $S_D(T)$ in the straightforward way, for cubic NLS with parameter β, as

$$S_D(T)u_0 = \int_0^T e^{i\beta\Delta(t-\tau)} u^2(\cdot,\tau)\bar{u}(\cdot,\tau)\ d\tau,$$

and we make a cruder estimate than that of [3]. We estimate the norm of this in the Sobolev space H^{s+n+1} :

$$\|S_D(T)u_0\|_{s+n+1} = \left\| \int_0^T e^{i\beta\Delta(T-\tau)} u^2(\cdot,\tau)\bar{u}(\cdot,\tau)\ d\tau \right\|_{s+n+1}$$
$$\leq \int_0^T \left\| e^{i\beta\Delta(T-\tau)} u^2(\cdot,\tau)\bar{u}(\cdot,\tau) \right\|_{s+n+1}\ d\tau.$$

We use the fact that the semigroup $e^{i\Delta t}$ gives an isometry on Sobolev spaces, and we use our doubling estimate; thus, we can estimate this as simply

$$\|S_D(T)u_0\|_{s+n+1} \leq T_2 \left(\sup_{\tau\in[0,T_2]} \|u(\cdot,\tau)\|_{s+n+1} \right)^3 \leq 8T_2\|u_0\|^3_{s+n+1}. \tag{17}$$

We now interpolate, using two of the factors of u_0 on the right hand side, making use of Corollary 2. For the third factor, we estimate it as $\|u_0\|_{s+n+1} \leq \|u_0\|_{s+2n+2}$. These considerations yield the existence of $d_2 > 0$ such that

$$\|S_D(T)u_0\|_{s+n+1} \leq d_2\|u_0\|_s\|u_0\|^2_{s+2n+2}.$$

This verifies that Assumption 4 holds for (1), with $\eta = r_1$, $p = n+1$, $\tilde{p} = 2n+2$ and $q = 2$.

Remark 4. *We remark that we could actually lower the value of $\tilde{p}$, by using all three factors of $\|u_0\|^3_{s+n+1}$ on the right-hand side of* (17) *for interpolation, rather than only using two. Similarly, if we instead considered, for example, a quintic nonlinear Schrödinger equation, the value of $\tilde{p}$ could be lowered. For the cubic equation, we estimate the terms as above for simplicity.*

3.4. **Proof of the main theorem.** We have now verified that Lemma 5 applies. Let $\bar{c} = d_1 d_2$; without loss of generality, we take $\bar{c} \geq 1$. Recall the definition of c_S as the Sobolev embedding constant, $c_S \geq 1$, in (3). We choose $r_2 > 0$ to satisfy

$$r_2 = \min\left\{ \frac{r_1}{2}, \frac{1}{\sqrt{c_S}\sqrt{\bar{c}+T_2c_S}\sqrt{33}} \right\}.$$

Let $T \in W_\delta$ be given. Let $v_0 \in H^{s+2n+2}$ be given such that v_0 is the initial data for a time-periodic solution of (1) with temporal period T, and such that $\|v_0\|_{s+2n+2} < r_2$. Since $r_2 < r_1$, the conclusion of Lemma 5 is

$$\|v_0 - \langle\!\langle v_0 \rangle\!\rangle\|_s \leq \bar{c}\|v_0\|_s\|v_0\|^2_{s+2n+2}. \tag{18}$$

Furthermore, for all $t \in [0,T_2]$, we have $\|v(\cdot,t)\|_{s+2n+2} \leq 2\|v_0\|_{s+2n+2} < 2r_2 \leq r_1$. Note that if v_0 is the initial data for a time-periodic solution of (1) with temporal period T, then so is $v(\cdot,t)$, for any t. By Lemma 5, this implies the following, for any t :

$$\|v(\cdot,t) - \langle\!\langle v(\cdot,t) \rangle\!\rangle\|_s \leq \bar{c}\|v(\cdot,t)\|_s\|v(\cdot,t)\|^2_{s+2n+2}. \tag{19}$$

Since we have taken v_0 to satisfy $\|v_0\|_{s+2n+2} < r_2$, we conclude the following:

$$\bar{c}\|v_0\|^2_{s+2n+2} \leq \frac{1}{33c_S}. \tag{20}$$

Combining this with (18), we find the following:

$$\|v_0 - \langle\langle v_0\rangle\rangle\|_s \leq \frac{\|v_0\|_s}{33c_S}. \tag{21}$$

Note that the constant function $\langle\langle v_0\rangle\rangle$ is an element of H^s, and the norm satisfies $\|\langle\langle v_0\rangle\rangle\|_s = |\langle\langle v_0\rangle\rangle|$. Using this fact, and using the reverse triangle inequality with (21), we find the following:

$$|\langle\langle v_0\rangle\rangle| \geq \frac{(33c_S - 1)\|v_0\|_s}{33c_S} \geq \frac{(33c_S - c_S)\|v_0\|_s}{33c_S} = \frac{32\|v_0\|_s}{33}. \tag{22}$$

That is, we see that the mass of v_0 must be substantially concentrated in its mean. Similarly, we use the doubling estimate with (19) and (20), to get a similar statement to (20):

$$\|v(\cdot,t) - \langle\langle v(\cdot,t)\rangle\rangle\|_s \leq \bar{c}\|v(\cdot,t)\|_s\|v(\cdot,t)\|_{s+2n+2}^2 \leq 8\bar{c}\|v_0\|_s\|v_0\|_{s+2n+2}^2 \leq \frac{8\|v_0\|_s}{33c_S}. \tag{23}$$

We are attempting to show that all values $v(x,t)$ must remain close to $\langle\langle v_0\rangle\rangle$ throughout the evolution. Our next step is to show that the mean of $v(\cdot,t)$ remains close to the mean of v_0. Using the evolution equation, (1), we have the following:

$$|\langle\langle v(\cdot,t)\rangle\rangle_t| = \frac{1}{\mathrm{vol}(\mathbb{T}^n)}\left|\int_{\mathbb{T}^n} |v(\cdot,t)|^2 v(\cdot,t)\ dx\right| \leq \|v(\cdot,t)\|_0^2 |v|_\infty \leq c_S\|v(\cdot,t)\|_s\|v(\cdot,t)\|_{s+2n+2}^2. \tag{24}$$

We have used Sobolev embedding for the last inequality. Recall as well that we have assumed, without loss of generality, that $\beta \in \{-1,1\}$. Using the doubling estimate and the fundamental theorem of calculus, (24) implies the following:

$$|\langle\langle v(\cdot,t)\rangle\rangle - \langle\langle v_0\rangle\rangle| \leq 8Tc_S\|v_0\|_s\|v_0\|_{s+2n+2}^2.$$

We then use our definition of r_2 to continue to rewrite this:

$$|\langle\langle v(\cdot,t)\rangle\rangle - \langle\langle v_0\rangle\rangle| \leq \frac{8\|v_0\|_s}{33c_S}. \tag{25}$$

We are now able to estimate the difference of $v(x,t)$ and $\langle\langle v_0\rangle\rangle$. Let x and t be given:

$$|v(x,t) - \langle\langle v_0\rangle\rangle| \leq |v(\cdot,t) - \langle\langle v_0\rangle\rangle|_\infty \leq c_S\|v(\cdot,t) - \langle\langle v_0\rangle\rangle\|_s. \tag{26}$$

We now estimate $\|v(\cdot,t) - \langle\langle v_0\rangle\rangle\|_s$, by adding and subtracting:

$$\|v(\cdot,t) - \langle\langle v_0\rangle\rangle\|_s \leq \|v(\cdot,t) - \langle\langle v(\cdot,t)\rangle\rangle\|_s + |\langle\langle v(\cdot,t)\rangle\rangle - \langle\langle v_0\rangle\rangle|. \tag{27}$$

We substitute the estimates (23) and (25):

$$\|v(\cdot,t) - \langle\langle v_0\rangle\rangle\|_s \leq \frac{16\|v_0\|_s}{33c_S}.$$

Using this with (26), we have

$$|v(x,t) - \langle\langle v_0\rangle\rangle| \leq \frac{16\|v_0\|_s}{33}, \tag{28}$$

where x and t are arbitrary.

What we have accomplished in demonstrating (28) is that we have shown that the values of the solution must all stay confined in a closed ball about $\langle\langle v_0\rangle\rangle$. In particular, for all x and t, we have $v(x,t) \in B\left(\langle\!\langle v_0 \rangle\!\rangle; \frac{16\|v_0\|_s}{33}\right)$. We now use Lemma 3. To do so, we need to compare the size of the center and the size of the radius of this ball; for this, we use (22). Letting $r = \frac{16\|v_0\|_s}{33}$ and $z = \langle\!\langle v_0 \rangle\!\rangle$, we have the following estimate:

$$r = \frac{16\|v_0\|_s}{33} = \frac{32\|v_0\|_s}{33 \cdot 2} < \frac{32\|v_0\|_s}{33 \cdot \sqrt{2}} \leq \frac{|\langle\!\langle v_0 \rangle\!\rangle|}{\sqrt{2}} = \frac{|z|}{\sqrt{2}}.$$

Now, assume (for the sake of contradiction) that $\langle\!\langle v_0 \rangle\!\rangle$ is nonzero. Thus, Lemma 3 applies, and this ball either does not intersect the real axis, or does not intersect the imaginary axis. That $v(x,t)$ is in this ball for all x and t, and that the ball does not intersect one of the axes implies that one of the following must hold: (a) $\mathrm{Re}(v(x,t)) > 0$ for all x and t, (b) $\mathrm{Re}(v(x,t)) < 0$ for all x and t, (c) $\mathrm{Im}(v(x,t)) > 0$ for all x and t, or (d) $\mathrm{Im}(v(x,t)) < 0$ for all x and t.

Without loss of generality, assume that option (a) holds, so that $\mathrm{Re}(v(x,t)) > 0$ for all x and t. We may infer the evolution equation for $\langle\!\langle v \rangle\!\rangle$ from (1), finding

$$\langle\!\langle v \rangle\!\rangle_t = \frac{\beta i}{\mathrm{vol}(\mathbb{T}^n)} \int_{\mathbb{T}^n} |v|^2 v \, dx.$$

We may then take the imaginary part of this, noticing that the imaginary part commutes with the time derivative:

$$\left(\mathrm{Im}\left\{\langle\!\langle v \rangle\!\rangle\right\}\right)_t = \frac{\beta}{\mathrm{vol}(\mathbb{T}^n)} \int_{\mathbb{T}^n} |v|^2 \mathrm{Re}\left\{v\right\} \, dx.$$

If $\beta > 0$, since we are assuming that choice (a) holds, this implies that $\left(\mathrm{Im}\left\{\langle\!\langle v \rangle\!\rangle\right\}\right)_t > 0$ for all t. Since $\langle\!\langle v \rangle\!\rangle_t$ must integrate to zero over time intervals of length T, this is a contradiction. The other cases, which include $\beta < 0$ or options (b), (c), or (d), are entirely similar.

We have reached a contradiction, and so we have shown that it must be the case that $\langle\!\langle v_0 \rangle\!\rangle = 0$. Keeping in mind Remark 3, and in particular considering (18) with $\langle\!\langle v_0 \rangle\!\rangle = 0$, we have established

$$\|v_0\|_s \leq \bar{c}\|v_0\|_s \|v_0\|_{s+2n+2}^2. \tag{29}$$

The condition $\|v_0\|_{s+2n+2} < r_2$ implies $\bar{c}\|v_0\|_{s+2n+2}^2 < 1$. Using this with (29), we see that $\|v_0\|_s = 0$. This completes the proof of the theorem.

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