GRAVITY PERTURBED CRAPPER WAVES

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ABSTRACT. Crapper waves are a family of exact periodic traveling wave solutions of the free-surface irrotational incompressible Euler equations; these are pure capillary waves, meaning that surface tension is accounted for but gravity is neglected. For certain parameter values, Crapper waves are known to have multi-valued height. Using the implicit function theorem, we prove that any of the Crapper waves can be perturbed by the effect of gravity, yielding the existence of gravity-capillary waves nearby to the Crapper waves. This result implies the existence of traveling gravity-capillary waves with multi-valued height. The solutions we prove to exist include waves with both positive and negative values of the gravity coefficient. We also compute these gravity perturbed Crapper waves by means of a quasi-Newton iterative scheme (again, using both positive and negative values of the gravity coefficient). A phase diagram is generated which depicts the existence of single-valued and multivalued traveling waves in the gravity-amplitude plane. A new largest water wave is computed which is composed of a string of bubbles at the interface.

1. Introduction

We study the irrotational, incompressible Euler equations for a fluid bounded above by a free surface, with vacuum above the free surface. We consider a fluid region which is infinitely deep in the vertical direction and periodic in the horizontal direction. We seek traveling wave solutions, or solutions for which the free surface is of permanent form and steadily translating. We consider the effect of surface tension at the fluid boundary.

For this problem, in the absence of gravity, a family of exact solutions is known; these are called Crapper waves, as they were discovered by Crapper [10]. There is exposition of these waves in the book of Crapper [11], and also, in more detail, in the book of Kinsman [27]. A formula for the Crapper waves will be given in Section 2 below.

Kinnersley extended the Crapper waves to the case of finite depth [26]. As in the infinite depth case that Crapper studied [10], the traveling waves are found as exact solutions, in this case involving elliptic functions. Building upon work by Tanveer for translating bubbles [43], Crowdy gives a different derivation of the Crapper waves, using conformal maps [13]. The formulation of Crowdy also allows for the identification of other, previously unknown, exact solutions of the free-surface Euler equations with surface tension. Crowdy further shows that exact steady free-surface Euler flows such as the Crapper waves yield, through a transformation, exact steady Hele-Shaw flows [12].

Uniqueness and stability of Crapper waves have also been studied. Assuming a certain positivity condition, Okamoto has proved uniqueness of the Crapper waves (that is, any traveling pure capillary water wave on infinite depth which satisfies the positivity condition is a Crapper wave) [34]; see also the discussion in the book of Okamoto and Shōji [35]. Tiron and Choi studied stability of the Crapper waves [44]. Stability was also studied by Hogan [21], and by Chen and Saffman [8].

Given an exact solution or family of exact solutions of a nonlinear partial differential equation, a natural question to ask is, Can we perturb these solutions to find other, nearby solutions? Since the Crapper waves are irrotational, pure capillary water waves, there are then several natural directions in which to perturb them: through the inclusion of gravity, through the inclusion of vorticity in the bulk of the fluid, and through the addition of an upper fluid to replace the vacuum above the waves. In this contribution, we prove that it is indeed possible to perturb any Crapper wave solution through the inclusion of gravity, either positive or negative. We do this by using a modified Liapunov-Schmidt analysis. We call the waves we find gravity perturbed Crapper waves. Perturbing the Crapper waves with vorticity or an additional fluid is expected to be the subject of future research.

In addition to proving they exist, we compute gravity perturbed Crapper waves and study the differences in features of the waves caused by the presence of gravity. The computational method used is the method introduced by the authors in [1]. This method uses a normalized arclength parameterization of the free surface, as developed for a numerical method for initial value problems for vortex sheets in [22], [23], and used analytically (again for the initial value problem) in [5].

A number of other papers have computed overturning traveling waves in free-surface Euler flows. Baker, Meiron, and Orszag formulated the traveling wave problem in such a way as to allow for overturning waves, but only computed waves with single-valued height in interfacial flows [7]. Other studies which computed interfacial waves with multi-valued height are those of Saffman and Yuen [39], Meiron and Saffman [31], Turner and Vanden-Broeck [46], and Grimshaw and Pullin [17].

The self-intersection and extreme forms of gravity-capillary waves have been studied previously. In particular, a detailed study was made by Hogan in [20]. The method of [20] is a boundary perturbation method, in the style of the expansions of Stokes (1847) and Wilton (1915) [9, 51]. Similar boundary perturbation methods have been applied to the water wave problem, with and without surface tension [36, 40, 38, 37, 33, 3]. As noted in [33], these methods can be susceptible to floating point instabilities. Hogan observed this problem in his results, and reacted by working in quadruple precision and restricting the number of terms in his series expansions. The Transformed Field Expansions method (of Nicholls and collaborators), a boundary perturbation algorithm, is stable to such errors [33, 3], however it cannot compute overturned traveling waves.

A popular alternative to boundary perturbation is the combination of Fourier collocation and a quasi-Newton solver. Such an approach has been applied to compute gravity-capillary waves in a variety of settings, including finite and infinite depth, periodic and solitary waves [16]. For overturning waves, the majority of these computations are based on a conformal mapping applied to the fluid domain. The closest works in this latter class to the present are those of Debiane and Kharif [14] and Debiane, Kharif and Amaouche [15], in which the largest traveling waves are computed for a sampling of Bond numbers, that of Vanden-Broeck and Schwartz [41], where a host of traveling waves are computed at various amplitudes and Bond numbers, and Vanden-Broeck and Keller [49] who computed the analogue of Crapper waves with larger bubbles (by varying the internal bubble pressure). Our numerical method also uses Fourier collocation and a quasi-Newton solver, but rather than being based on a conformal mapping, it instead parameterizes the interface by arclength [1]. This allows us to keep a uniform grid spacing along the interface, as opposed to having our grid points chosen by the mapping.

To complement our existence results, we numerically compute wave profiles and speeds, varying both gravity and amplitude. Because the existence proof relies on the implicit function theorem, it does not require that gravity be positive; therefore, in our computations, we perturb the Crapper wave with both positive and negative gravity. We compute continuous branches of traveling waves connecting Crapper waves ($g = 0$) to gravity-capillary waves with $g = 20$, as well as to waves with negative gravity $0 \leq g <$ -1. Unlike methods based on amplitude expansions, our method need not be altered to compute the resonant Wilton ripples - see [18, 4]. Our results support the conclusions of Schwartz and Vanden-Broeck [41]; we observe traveling waves at the Wilton ripple Bond numbers which are continuously embedded amongst the traveling waves computed elsewhere.

For the branches of waves bifurcating from the $g = 0$ Crapper waves which we compute, as gravity increases, we observe that the Crapper wave is continuously connected to *solitary* gravity-capillary waves. These computations serve as numerical verification of the argument of Longuet-Higgins, whereby asymptotics about the Crapper wave are used to approximate gravity-capillary solitary waves [29]. For the branches of waves with $g < 0$, we observe that the large amplitude limit is typically a stationary (or standing) wave, rather than a traveling wave with a self-intersecting profile. We have generated a phase portrait of the existence of multi-valued and single-valued traveling waves in the gravity-amplitude plane. A new, globally largest water wave is computed which is both standing and self-intersecting. This wave consists of a string of bubbles and droplets at the fluid interface.

The paper is organized as follows: in Section 2, we describe the Crapper waves. In Section 3, we give our proof, using the implicit function theorem, to show the existence of gravity perturbed Crapper waves. In Section 4 we describe our numerical method and give our numerical results.

2. The Crapper Waves

In Section 1.1 of the book of Okamoto & Shōji [35], they give the following equation whose solutions give traveling spatially periodic gravity-capillary waves in a two dimensional fluid of infinite depth:

$$F(\theta;p,q) := e^{2H\theta}\frac{dH\theta}{da} - pe^{-H\theta}\sin(\theta) + q\frac{d}{da}\left(e^{H\theta}\frac{d\theta}{da}\right) = 0. \tag{1}$$

Here the independent variable a is in $\mathbf{T} := [-\pi, \pi]$, θ is the tangent angle to the surface and satisfies periodic boundary conditions, and H is the periodic Hilbert transform:

$$Hf(a) := \frac{1}{2\pi}\text{PV}\int_{-\pi}^{\pi}\cot\left(\frac{1}{2}(a-s)\right)f(s)ds.$$

For background on the periodic Hilbert transform, the interested reader could consult, for instance, [19]. The non-dimensional constants p and q are given by

$$p := \frac{gL}{2\pi c^2} \quad \text{and} \quad q := \frac{2\pi\tau}{\rho c^2 L},$$

where L is the spatial period of the wave, c is the traveling wave speed, g is the acceleration due to gravity and τ is the surface tension constant. We remark that in the sequel, we will always take q to be positive. This formulation of the traveling water wave problem follows from complex variable methods initiated by Stokes [42], with important contributions from Levi-Civita [28].

An important feature here is that the free surface is not parameterized by arclength; reconstruction of the surface from θ requires some more information. Specifically, if $\theta(a)$ is a solution of (1), the parameterization of the free surface $(x(a), y(a))$ is given by:

$$\begin{aligned} \frac{dx}{da} &:= -\frac{L}{2\pi} e^{-H\theta(a)} \cos(\theta(a)) \\ \frac{dy}{da} &:= -\frac{L}{2\pi} e^{-H\theta(a)} \sin(\theta(a)). \end{aligned} \tag{2}$$

If one sets $p = 0$, one can find exact formulae for solutions θ of (1), which are called *Crapper waves*. We briefly explain these solutions now. When $p = 0$, (1) becomes:

$$e^{2H\theta} \frac{dH\theta}{da} + q \frac{d}{da}\left(e^{H\theta} \frac{d\theta}{da}\right) = 0.$$

The right hand side is a perfect derivative:

$$\frac{d}{da}\left(\frac{1}{2} e^{2H\theta} + q e^{H\theta} \frac{d\theta}{da}\right) = 0.$$

Integrating gives:

$$\frac{1}{2} e^{2H\theta} + q e^{H\theta} \frac{d\theta}{da} = \text{cst}. \tag{3}$$

In [35], it is shown that:

Lemma 1. *For any $\theta \in L^2(\mathbf{T})$,*

$$\int_{-\pi}^{\pi} e^{\pm H\theta(a) - i\theta(a)} da = 2\pi. \tag{4}$$

With this, one can show that the constant in (3) is $1/2$. The equation (3) can then be rewritten as

$$q \frac{d\theta}{da} + \sinh(H\theta) = 0. \tag{5}$$

Take $q \geq 1$ and let A be either of the solutions of:

$$q = \frac{1 + A^2}{1 - A^2}.$$

Note that $A \in (-1, 1)$. Let

$$\omega(z) := 2i \log\left(\frac{1 + Az}{1 - Az}\right),$$

and

$$\theta_q(a) := \Re\omega(e^{ia}).$$

Then we have:

Theorem 2. (Crapper)

$$q\frac{d\theta_q}{da} + \sinh(H\theta_q) = 0.$$

Note that $\theta_q(a)$ is an odd function of a. The functions θ_q gives the Crapper wave.

Linearizing (5) about θ_q gives the operator:

$$\Gamma u := qu' + \cosh(H\theta_q)Hu.$$

(Here and below, the prime indicates differentiation with respect to a.) The following Lemma, which is equivalent to Lemma 2.1 in [35], is used there to prove that the Crapper waves are locally unique:

Lemma 3. (Okamoto & Shōji) *For $q > 1$, zero is an eigenvalue of $\Gamma : X^1 \to L^2(\mathbf{T})$ with geometric multiplicity one and algebraic multiplicity two. The eigenfunction is $\dfrac{d\theta_q}{da}$ and the generalized eigenfunction is $\dfrac{\partial\theta_q}{\partial q}$.*

In the above,

$$X^s := \{f \in H^s(\mathbf{T}) : (f, 1) = 0\}$$

and

$$(f, v) := \int_{-\pi}^{\pi} f(a)v(a)da.$$

In this article we show that Crapper waves perturb to solutions of (1) when $p \sim 0$. In particular, we prove the following theorem:

Theorem 4. *For all $q > 1$, there exist $P = P(q) > 0$ and a C^∞ function*

$$\Theta_q : (-P, P) \longrightarrow X^2$$

such that $F(\Theta_q(p); p, q) = 0$ for $|p| < P$ and $\Theta_q(0) = \theta_q$. Moreover, $\Theta_q(p)$ is an odd function of a, and Θ_q is smooth with respect to q.

Remark 1. *Notice that this theorem gives the existence of traveling gravity-capillary waves nearby to the Crapper waves, for small values of p, with either $p > 0$ or $p < 0$, i.e., with either positive or negative values of the acceleration due to gravity. There are some theorems in the literature on the non-existence of traveling water waves with negative gravity, in the case of pure gravity waves* [25], [45].

The method of proof is, roughly speaking, a Liapunov-Schmidt analysis, and follows relatively quickly from Theorem 2 and Lemma 3. Despite the brevity of its proof, we point out a particularly important and novel feature of Theorem 4: it implies the existence of periodic traveling capillary-gravity waves which overhang. It is well-known that for q large enough the profile of the Crapper wave is not given by a function over the horizontal Eulerian coordinate. Since our theorem applies for any $q > 1$ and Θ_q is continuous with respect to p, we have the same feature for the gravity perturbed Crapper wave.

We will be employing the implicit function theorem several times in our proof, and for completeness we state the version of this theorem we use here:

Theorem 5. (The Implicit Function Theorem) *X, Y and Z are Banach spaces and $\zeta : X \times Y \to Z$ is C^k, $k \geq 1$. If $\zeta(x^*, y^*) = 0$ and $D_x\zeta(x^*, y^*)$ is a bijection from X to Z then there exists $\epsilon > 0$ and a unique C^k map $\chi : Y \to X$ such that $\chi(y^*) = x^*$ and $\zeta(\chi(y), y) = 0$ when $\|y - y^*\|_Y < \epsilon$.*

Statements of implicit function theorems can be found in many standard texts; for example, a similar statement to Theorem 5 can be found in Chapter 13 of [24].

3. Existence of Gravity Perturbed Crapper Waves

Our first task is to rewrite the problem (1) as a perturbation of (5). Notice that we can write (1) as:

$$\frac{d}{da}\left[\frac{1}{2}e^{2H\theta} + qe^{H\theta}\theta'\right] - pe^{-H\theta}\sin(\theta) = 0.$$

Integrating this from 0 to a we get:

$$\frac{1}{2}e^{2H\theta} + qe^{H\theta}\theta' - p\int_0^a e^{-H\theta(a')}\sin(\theta(a'))da' = \gamma, \tag{6}$$

where γ is a constant. Note that it is possible to determine the value of γ in advance for a solution, as is done in the derivation of (5) above. However, it is to our advantage in the proof to leave this constant undetermined at this time as it gives us an extra parameter at our disposal. Recalling the constant of integration when $p = 0$ was $1/2$, we set

$$\gamma = \frac{1}{2} + \kappa,$$

where $\kappa \in \mathbf{R}$. Also, we define

$$I(\theta)(a) := \int_0^a e^{-H\theta(a')}\sin(\theta(a'))da'.$$

We then recast (6) as:

$$\frac{1}{2}e^{2H\theta} + qe^{H\theta}\theta' - pI(\theta) - \frac{1}{2} - \kappa = 0.$$

A quick rearrangement of terms converts this to:

$$\Phi(\theta; p, \kappa) := q\theta' + \sinh(H\theta) - pe^{-H\theta}I(\theta) - \kappa e^{-H\theta} = 0. \tag{7}$$

Observe that Theorem 2 shows that $\Phi(\theta_q; 0, 0) = 0$. Moreover, if we linearize Φ at $\theta = \theta_q$ and $p = \kappa = 0$, we have:

$$D_\theta \Phi(\theta_q; 0, 0) = \Gamma.$$

Lemma 3 tells us that Γ is not invertible. As such we cannot directly employ the implicit function theorem to find solutions of $\Phi = 0$.

Nevertheless, we can eliminate the kernel of Γ by restricting attention to odd functions. Let

$$X^1_{odd} := \left\{ f \in H^1(\mathbf{T}) : f \text{ is odd} \right\},$$

and

$$Y^0_{even} := \left\{ f \in L^2(\mathbf{T}) : f \text{ is even} \right\}.$$

Of course X^1_{odd} is a subspace of X^1 and Y^0_{even} is a subspace of $L^2(\mathbf{T})$. Then

Proposition 1. *$\Phi(\theta; p, \kappa)$ is a C^∞ map from $X^1_{odd} \times \mathbf{R}^2$ into Y^0_{even}.*

We omit the proof of this. The key thing is that d/da, H and I all map odd functions to even ones.

Recall that θ_q is odd and thus θ_q' is even. Therefore the kernel of Γ as a map on X^1_{odd} is trivial. That is, we have the following corollary of Lemma 3:

Corollary 6. *When Γ is viewed as map from X^1_{odd} to Y^0_{even},*

$$\ker \Gamma = \{0\}.$$

It might at this time appear that we could apply the implicit function theorem to find solutions of (7). However this is not the case. While Γ is injective as a map from X^1_{odd} to Y^0_{even}, it is not surjective, as the following proposition demonstrates:

Proposition 2. *Let $f \in Y^0_{even}$. Then there exists $u \in X^1_{odd}$ with*

$$\Gamma u = f$$

if and only if

$$(f, \cos(\theta_q)) = 0.$$

Proof. Lemma 1 implies, for any function $w \in L^2(\mathbf{T})$,

$$\int_{-\pi}^{\pi} e^{-iw(a)} \sinh(Hw(a)) da = 0.$$

Taking the real part of this gives:

$$\int_{-\pi}^{\pi} \cos(w(a)) \sinh(Hw(a)) da = 0.$$

Also, since $\dfrac{d}{da} \sin(w(a)) = \cos(w(a)) w'(a)$, we have $\displaystyle\int_{-\pi}^{\pi} \cos(w(a)) w'(a) da = 0$. So, for any function $w \in H^1(\mathbf{T})$:

$$\int_{-\pi}^{\pi} \cos(w(a)) \left[\sinh(Hw(a)) + q w'(a)\right] da = 0. \tag{8}$$

Fix $u \in H^1(\mathbf{T})$ and let $w = \theta_q + \epsilon u$, where $\epsilon \in \mathbf{R}$. Inserting this into (8) gives, for all ϵ:

$$\int_{-\pi}^{\pi} \cos(\theta_q(a) + \epsilon u(a)) \left[\sinh\left(H\theta_q(a) + \epsilon H u(a)\right) + q\theta_q'(a) + \epsilon q u'(a)\right] da = 0.$$

Differentiating this with respect to ϵ and then setting $\epsilon = 0$ gives:

$$-\int_{-\pi}^{\pi} \sin(\theta_q(a))u(a)\left[\sinh(H\theta_q(a)) + q\theta_q'(a)\right] da$$
$$+\int_{-\pi}^{\pi} \cos(\theta_q(a))\left[\cosh(H\theta_q(a))Hu(a) + qu'(a)\right] da = 0.$$

The first line vanishes by Theorem 2. The term in square brackets in the second line is Γu. Thus we have, for all $u \in H^1(\mathbf{T})$:

$$\int_{-\pi}^{\pi} \cos(\theta_q(a))\Gamma u(a)da = (\Gamma u, \cos(\theta_q)) = 0. \tag{9}$$

If $f \in Y^0_{even}$ and $u \in X^1_{odd}$ satisfy $\Gamma u = f$, then (9) tells us

$$(f, \cos(\theta_q)) = 0.$$

And so we have shown the "only if" part of the Proposition.

To finish the proof, we need to show that if $f \in Y^0_{even}$ and $(f, \cos(\theta_q)) = 0$, then there exists $u \in X^1_{odd}$ with $\Gamma u = f$. First, note that

$$Ku := \cosh(H\theta_q)Hu$$

defines a bounded map from X^1_{odd} to Y^0_{even}. It is also bounded from $L^2(\mathbf{T})$ to itself. Suppose that $\{u_n\} \subset X^1_{odd}$ is a bounded sequence. Since $X^1_{odd} \subset H^1(\mathbf{T}) \subset\subset L^2(\mathbf{T})$ (due to the Rellich-Kondrachov theorem) we know $\{u_n\}$ contains a subsequence which converges in $L^2(\mathbf{T})$. We abuse notation and call this subsequence $\{u_n\}$. Since the functions u_n are odd, so is the limit. Also, K is bounded on $L^2(\mathbf{T})$ and thus $\{Ku_n\}$ is a convergent sequence of even functions in $L^2(\mathbf{T})$. The limit is even. Thus K is a compact operator from X^1_{odd} to Y^0_{even}.

Now, $\Gamma u = qu' + K$ which means that Γ is a compact perturbation of

$$\Gamma_0 := q\frac{d}{da}.$$

We claim that Γ_0 is a Fredholm operator from X^1_{odd} to Y^0_{even} with index -1. This is straightforward. We recall that q is nonzero, and that the kernel of d/da is the constant functions. The only odd constant function is 0. Therefore the kernel of Γ_0 in X^1_{odd} is trivial. Likewise, $\Gamma_0 u = f$ has a periodic solution u if and only if $\int_{-\pi}^{\pi} f(a')da' = 0$. This implies that the cokernel of Γ_0 is one dimensional. Thus the index of Γ_0 is -1.

Since compact perturbations do not change the index of an operator [24], we know that Γ is also a Fredholm operator from X^1_{odd} to Y^0_{even} with index -1. Since we know the kernel of Γ is trivial from Corollary 6, we must therefore have that the dimension of the cokernel of Γ is equal to 1.

Since $(\cos(\theta_q), \cos(\theta_q)) \neq 0$, the work above tells us that $\cos(\theta_q)$ is not in the range of Γ. Thus the equivalence class of this function, which we call $[c]$, spans the cokernel of Γ.

Now take $f \in Y^0_{even}$ with $(f, \cos(\theta_q)) = 0$. Let $[f]$ be the equivalence class of f in the cokernel. There must be a constant $\beta \in \mathbf{R}$ such that $[f] = \beta[c]$. This means that there exists $u \in X^1_{odd}$ with $\Gamma u = f - \beta \cos(\theta_q)$. From the first part of this proof, we know that this implies $(f - \beta\cos(\theta_q), \cos(\theta_q)) = 0$ which implies $\beta = 0$. This in turn implies that $\Gamma u = f$ and we are done.

□

So Γ is not surjective. Our method is a modification of the Liapunov-Schmidt strategy commonly used in perturbation theory when the linearization of the problem fails to be injective. Define

$$\Pi u := \mu(\cos(\theta_q), u)\cos(\theta_q)$$

where $\mu := (\cos(\theta_q), \cos(\theta_q))^{-1}$. A quick calculation shows $\Pi^2 = \Pi$, and so Π is a projection. Moreover, (9) tells us that

$$\Pi\Gamma = 0. \tag{10}$$

Let $R := \ker \Pi$ and $M := R^\perp$. Proposition 2 tells us that $R = \Gamma(X^1_{odd})$. Let

$$\Phi_{red}(\theta; p, \kappa) = (1 - \Pi)\Phi(\theta; p, \kappa).$$

Note that $\Pi\Phi_{red} = (\Pi - \Pi^2)\Phi = 0$, and so Φ_{red} is a map from $X^1_{odd} \times \mathbf{R}^2$ into R.

From Theorem 2 we know

$$\Phi_{red}(\theta_q; 0, 0) = 0.$$

Also

$$D_\theta \Phi_{red}(\theta_q; 0, 0) = (1 - \Pi)\Gamma.$$

Since $\Pi\Gamma = 0$, we see that $(1 - \Pi)\Gamma$ is injective because Γ is. Moreover, by construction, $(1 - \Pi)\Gamma$ is surjective onto R. And thus we can, at last, apply the implicit function theorem: there exists a unique C^∞ map:

$$\Xi(p, \kappa) : \mathbf{R}^2 \to X^1_{odd},$$

so that

$$\Phi_{red}(\Xi(p, \kappa); p, \kappa) = (1 - \Pi)\Phi(\Xi(p, \kappa); p, \kappa) = 0$$

for all (p, κ) sufficiently small and

$$\Xi(0, 0) = \theta_q.$$

Now set

$$g(p, \kappa) := \frac{\Pi\Phi(\Xi(p, \kappa); p, \kappa)}{\mu\cos(\theta_q)} = (\cos(\theta_q), \Phi(\Xi(p, \kappa); p, \kappa)).$$

This is a map from $\mathbf{R}^2$ to itself. That is to say, $g(p, \kappa)$ satisfies $\Pi\Phi(\Xi(p, \kappa); p, \kappa) = \mu g(p, \kappa)\cos(\theta_q)$. If we find $g(p^*, \kappa^*) = 0$ then notice that $\Xi(p^*, \kappa^*)$ has:

$$\Phi(\Xi(p^*, \kappa^*); p^*, \kappa^*) = \Pi\Phi(\Xi(p^*, \kappa^*); p^*, \kappa^*) + (1 - \Pi)\Phi(\Xi(p^*, \kappa^*); p^*, \kappa^*) = 0.$$

That is to say, it solves (7).

Clearly $g(0, 0) = 0$. We claim that

$$D_\kappa g(0, 0) = -2\pi. \tag{11}$$

If so, we can call again on the implicit function theorem. This tells us that there exists a unique map, $\xi(p)$, smooth, with $\xi(0) = 0$ and defined for p sufficiently small, for which

$$g(p, \xi(p)) = 0.$$

Let

$$\Theta_q(p) := \Xi(p, \xi(p)).$$

This is the map whose existence we were hoping to establish. There are two things left to check. The first is to establish (11). The second is to show that $\Theta_q(p) \in X^2$, not just X^1_{odd}. This latter is important in that a solution of (7) in X^2 will also be a solution of (1), which is our ultimate goal. We mention that the smoothness of Θ_q with respect to q follows from our repeated use of the implicit function theorem.

For (11), note that

$$D_\kappa g(0,0) = \frac{\Pi\left[\Gamma\Xi_\kappa(0,0) + D_\kappa\Phi(\theta_q; 0, 0)\right]}{\mu\cos(\theta_q)}.$$

Since $\Pi\Gamma = 0$, we have:

$$D_\kappa g(0,0) = \frac{\Pi D_\kappa\Phi(\theta_q; 0, 0)}{\mu\cos(\theta_q)}.$$

Then, we see from (7) that $D_\kappa\Phi(\theta_q; 0, 0) = -e^{-H\theta_q}$, and therefore

$$\begin{aligned} D_\kappa g(0,0) = -\frac{\Pi e^{-H\theta_q}}{\mu\cos(\theta_q)} = -(e^{-H\theta_q}, \cos(\theta_q)) &= -\int_{-\pi}^{\pi} e^{-H\theta_q(a)}\cos(\theta_q(a))da \\ &= -\frac{1}{2}\int_{-\pi}^{\pi} e^{-H\theta_q(a)+i\theta_q(a)}da - \frac{1}{2}\int_{-\pi}^{\pi} e^{-H\theta_q(a)-i\theta_q(a)}da. \end{aligned}$$

Using Lemma 1, we have $D_\kappa g(0,0) = -2\pi$ as claimed.

Finally, let $\theta := \Theta_q(p)$. We know that $\theta \in X^1_{odd} \subset H^1(\mathbf{T})$ and that θ solves (7). Since $\theta \in H^1(\mathbf{T})$ we know $H\theta \in H^1(\mathbf{T})$ as well. This in turn implies that $\sinh(H\theta)$ and $e^{-H\theta}$ are in $H^1(\mathbf{T})$. Likewise for $I(\theta)$. Thus

$$\rho := \sinh(H\theta) - pe^{-H\theta}I(\theta) - \kappa e^{-H\theta} \in H^1(\mathbf{T}).$$

It is straightforward to see that since θ depends smoothly on p, so must ρ. Since (7) states that $\theta' = -\frac{1}{q}\rho$, we have $\theta' \in H^1(\mathbf{T})$ and thus $\theta \in H^2(\mathbf{T})$. Therefore we can differentiate (7) to see that θ solves (1), and depends smoothly on p. (We remark that this argument could be repeated to conclude higher regularity of θ.) This completes the proof of Theorem 4.

4. Numerical Results

In this section we augment the preceding existence proof with numerical computations of branches of traveling waves bifurcating from the Crapper waves. The branches are computed in the arclength parameterized vortex sheet formulation of the water wave problem [1]. Traveling waves are computed at different values of total displacement $h = \max(y) - \min(y)$ and gravity g. The branches are continuous in both parameters. We observe that the amplitude along a branch is either limited topologically, by a profile which self-intersects, or by a turning point, in which case the largest profile is a stationary (i.e., standing) wave. As

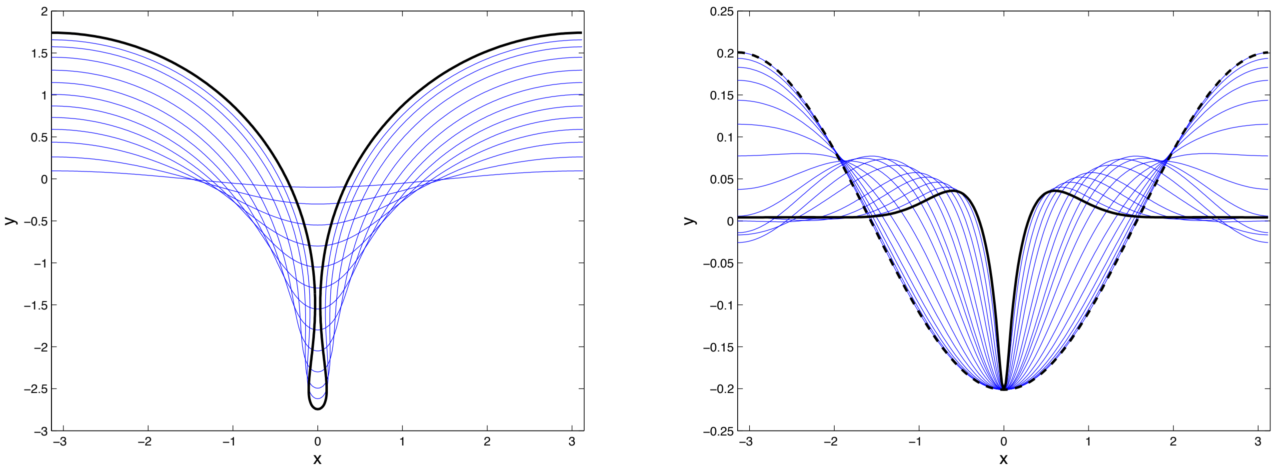

FIGURE 1. LEFT: Crapper waves with $g = 0$, at different displacements. The largest computed wave has scaled displacement $h/2\pi = 0.723$. RIGHT: Traveling wave profiles with $\|y\|_\infty = 0.2$ for different values of $g \in$ [-0.95, 20] with $\tau = 1$. The wave profiles deform continuously from the sinusoidal wave with $g =$ -0.95, marked with a dashed curve, toward a gravity-capillary solitary wave, similar to the profile with $g = 20$, marked with a thick solid line.

our preceding proof does not require positive values of gravity we compute perturbations of the Crapper wave with both signs of g. We allow large departures from $g = 0$, computing wave profiles with $g \in$ (-1, 20], in which interval we observe continuous dependence on gravity including the Wilton ripple configurations [51, 47, 30].

The numerical method used to compute these waves is based on the method of [1]. In this method, we find functions θ and γ which solve the equations

$$U = -c\sin(\theta), \qquad U_t = 0;$$

this is the traveling wave ansatz, as developed in [1]. Here, U is the normal velocity of the fluid interface, and is equal to the normal component of the Birkhoff-Rott integral (which will be introduced in the following paragraph). A Fourier-collocation method is used to discretize spatial derivatives. We seek symmetric, even profiles in terms of tangent angle θ and vortex sheet strength γ. The frame of the wave is defined so that the mean of γ and θ are zero. When the system is discretized with N spatial points, we must then solve for N Fourier modes and the wave speed c. The projection of the partial differential equation into Fourier space gives N equations, to which we append an equation fixing the amplitude to close the system. The resulting nonlinear system of algebraic equations is then solved via Broyden's method, and continuation in amplitude or gravity, similar to [2, 48].

The collocation method is standard, with the possible exception of the implementation of the Birkhoff-Rott integral,

$$\Phi(\mathbf{W})^* = \frac{1}{4\pi i} PV \int_0^{2\pi} \gamma(\beta) \cot\left(\frac{1}{2}(z(\alpha) - z(\beta))\right) \, d\beta,$$

which is computed using the method used in [1, 6], wherein the integral is split into singular and regular parts. The singular portion is $\frac{1}{2}H(\gamma/z_\alpha)$, where H is the Hilbert transform, and is computed using its definition in Fourier space, $\widehat{H(f)}(k) = -i\text{sign}(k)\hat{f}(k)$. The remainder integral, $\Phi(\mathbf{W})^* - \frac{1}{2}H(\frac{\gamma}{z_\alpha})$, is nonsingular and is computed using the trapezoidal rule at alternating grid points. The numerical results presented here use $n = 512$ spatial points, resulting in an arclength step size of $\Delta\alpha \approx 0.01$. At this resolution, the largest computed waves, which have overturned crests and are near pinch-off, have Fourier modes which decay to 10^{-8} by wavenumber $k = 256$.

We observe that traveling waves depend continuously on gravity. The meaningful physical quantity here is the Bond number $\sigma = \frac{\tau k^2}{g}$, which we study by fixing $k = 1, \tau = 1$ and varying g. The waves are also continuous in amplitude, and we use this fact to continue in both parameters to find the largest amplitude waves. Since our numerical method is based on a vortex sheet formulation, we cannot compute waves which self-intersect, like those in [49, 29] (that is, for self-intersecting waves, we would be unable to compute the Birkhoff-Rott integral). We are able to compute waves very close to the first self intersecting configuration, which we refer to as pinch off. The Crapper wave first self intersects when the scaled displacement $h/2\pi = (\max(y) - \min(y))/2\pi = 0.73$ [44, 10]. We have computed overturned profiles with scaled displacements up to $h/2\pi = 0.723$, or 99% of the Crapper pinch off displacement. Computed Crapper wave profiles at a sampling of amplitudes are in the left panel of Figure 1.

An advantage of quasi-Newton based methods over amplitude expansions, like Hogan [20] or Boyd and Haupt [18], is that no modification is necessary to compute traveling waves at the Wilton ripple resonant configurations. We observe Wilton ripples which are continuously embedded in the branches of traveling waves. In the right panel of Figure 1, the profiles of traveling waves are presented for $\|y\|_\infty = 0.2$ at a sampling of gravity values $g \in [\,-0.95, 20]$.

We have employed continuation schemes to follow continuous branches of traveling waves in both amplitude and gravity. Beginning with the Crapper wave, we observe that waves becomes narrower with increased gravity. As g increases, profiles become more solitary than periodic, gaining oscillatory tails similar those computed in [2, 32]. These changes occur at all amplitudes, see Figure 1 for the case of $||y||_\infty = 0.2$, however the size of oscillations in the tail decreases with amplitude. These numerical results support the argument of Longuet-Higgins [29], that the large amplitude Crapper wave is connected continuously to gravity-capillary solitary waves.

In Figure 2, we report both the globally largest wave and a phase portrait of the existence of traveling waves in the gravity-displacement plane. We observe that branches of traveling waves culminate in one of two ways: either in a standing wave or a self-intersecting profile. We consider self-intersecting interfaces to be the terminus of a branch of traveling waves. It is possible that branches of traveling waves continue to exist after self-intersection, however such profiles need to be computed with an alternative method, for example that of [49]. In the right panel of Figure 2, we report a phase portrait of the space of traveling waves. In the gray region, traveling waves are single valued functions of the x-coordinate. In the black region, the waves have overturned crests. The boundary between the black and white region is an estimate of the displacement of the largest wave. This boundary corresponds to

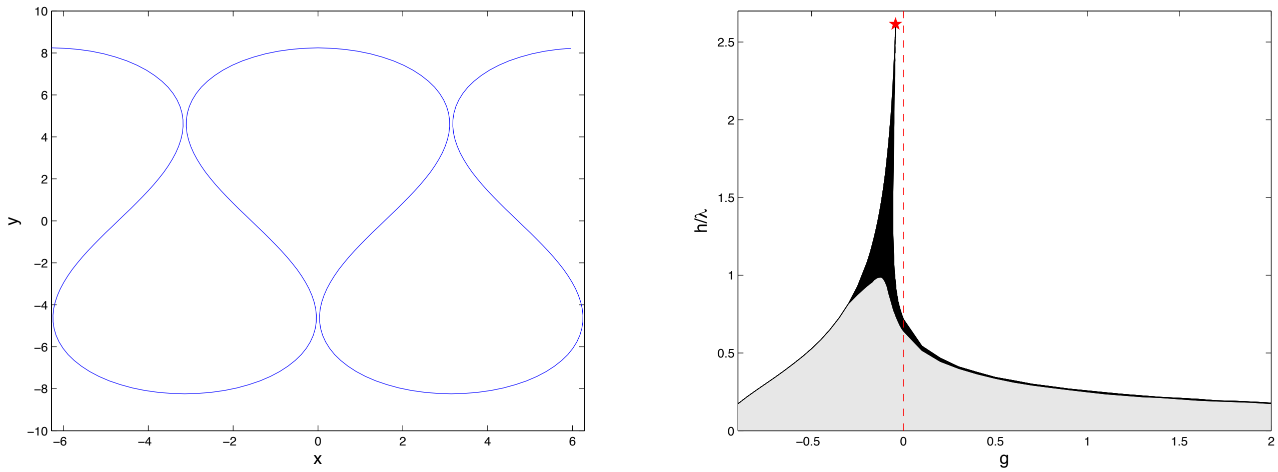

FIGURE 2. LEFT: Two periods of the profile of the largest computed water wave interface are plotted as a function of the horizontal coordinate. This wave is marked by a star in the right panel. This wave has $g \approx -.0432$, and $h/\lambda \approx 2.62$, more than triple that of the first self-intersecting Crapper wave. (Here, $\lambda = 2\pi$ is the horizontal period.) RIGHT: A phase portrait displays the regions of gravity-displacement space where single valued (gray) traveling waves and multivalued (black) traveling waves exist. The boundary with the white region is either a self-intersecting wave, to the right of the star, or a standing wave, to the left of the star. The globally largest wave is both standing and self-intersecting and is marked by the star.

self-intersection of the wave profile to the right of the star, and a standing wave ($c = 0$) to the left of the star. The globally largest wave is marked by a star, and is both stationary and self-intersecting.

That traveling waves exist with negative values of gravity is not surprising when one examines the linear situation. The phase speed of linear waves is,

$$c_p = \pm\sqrt{\tau|k| + \frac{g}{|k|}}.$$

One should expect traveling waves whenever the phase speed is real. In our numerical experiments, $k = 1$ and $\tau = 1$, so linear theory predicts traveling waves for $g >$ -1. When $g =$ -1, the speed of the wave is zero and we are left with stationary, standing waves. This is exactly what we observe numerically. Moreover, we observe that for most negative values of gravity the branches of nonlinear traveling waves have amplitudes limited by standing waves. These standing waves form the boundary of white region in the right panel of Figure 2, from $g =$ -1 to $g \approx$ -0.0432 marked by the star. Profiles of these standing waves are in the left panel of Figure 3. The water wave problem has a symmetry between waves moving to the left and right. Given a wave of a particular amplitude and speed, there is a corresponding wave with the same amplitude and negative speed. At stationary waves these two branches

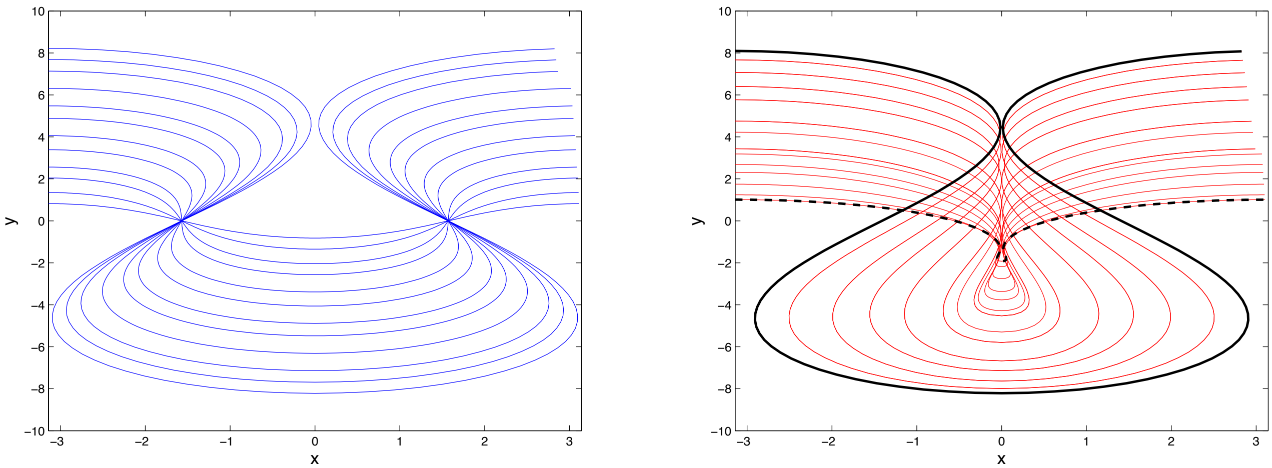

Figure 3. LEFT: The largest computed waves to the left of the star in Figure 2, with $g \in (-1, -0.0432)$. These profiles are all standing waves. RIGHT: The largest computed waves to the right of the star in figure 2, for $g \in (-0.0432, 0.2]$. In this range, the largest traveling wave profile self intersects, entraining a bubble at its trough. Both the length and breadth of the bubble are decreasing functions of gravity. The largest profile, marked with the thicker solid black line, is marked with the star in figure 2 and has $g \approx$ -0.0432. The smallest profile has $g = 0.2$, and is marked by the black dashed line.

of traveling waves intersect to form a closed loop. Whenever branches of traveling waves are thus connected, the standing wave will be the profile of limiting amplitude.

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References

[1] B. Akers, D.M. Ambrose, and J.D. Wright. Traveling waves from the arclength parameterization: Vortex sheets with surface tension. *Interfaces and Free Boundaries*, 2013. Accepted.

[2] B. Akers and P.A. Milewski. Dynamics of three-dimensional gravity-capillary solitary waves in deep water. *SIAM Journal on Applied Mathematics*, 70(7):2390–2408, 2010.

[3] Benjamin Akers and David P Nicholls. Traveling waves in deep water with gravity and surface tension. *SIAM Journal on Applied Mathematics*, 70(7):2373–2389, 2010.

[4] Benjamin F Akers and Wenxuan Gao. Wilton ripples in weakly nonlinear model equations. *Commun. Math. Sci*, 10(3):1015–1024, 2012.

[5] D.M. Ambrose. Well-posedness of vortex sheets with surface tension. *SIAM J. Math. Anal.*, 35(1):211–244 (electronic), 2003.

[6] D.M. Ambrose and J. Wilkening. Computation of symmetric, time-periodic solutions of the vortex sheet with surface tension. *Proceedings of the National Academy of Sciences*, 107(8):3361–3366, 2010.

[7] G.R. Baker, D.I. Meiron, and S.A. Orszag. Generalized vortex methods for free-surface flow problems. *J. Fluid Mech.*, 123:477–501, 1982.

[8] B. Chen and P.G. Saffman. Three-dimensional stability and bifurcation of capillary and gravity waves on deep water. *Stud. Appl. Math.*, 72(2):125–147, 1985.

[9] Alex DD Craik. George Gabriel Stokes on water wave theory. *Annu. Rev. Fluid Mech.*, 37:23–42, 2005.

[10] G.D. Crapper. An exact solution for progressive capillary waves of arbitrary amplitude. *J. Fluid Mech.*, 2:532–540, 1957.

[11] G.D. Crapper. *Introduction to water waves*. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester, 1984.

[12] D.G. Crowdy. Hele-Shaw flows and water waves. *Journal of Fluid Mechanics*, 409:223–242, 3 2000.

[13] D.G. Crowdy. A new approach to free surface Euler flows with capillarity. *Stud. Appl. Math.*, 105(1):35–58, 2000.

[14] M Debiane and C Kharif. A new limiting form for steady periodic gravity waves with surface tension on deep water. *Physics of Fluids*, 8:2780, 1996.

[15] Mohammed Debiane, Christian Kharif, and Mustapha Amaouche. A new method for the calculation of steady periodic capillary-gravity waves on water of arbitrary uniform depth. *European Journal of Mechanics-B/Fluids*, 19(6):855–870, 2000.

[16] Frédéric Dias and Christian Kharif. Nonlinear gravity and capillary-gravity waves. *Annual review of fluid mechanics*, 31(1):301–346, 1999.

[17] R.H.J. Grimshaw and D.I. Pullin. Extreme interfacial waves. *Phys. Fluids*, 29:2802–2807, 1986.

[18] Sue Ellen Haupt and John P Boyd. Modeling nonlinear resonance: A modification to the Stokes' perturbation expansion. *Wave Motion*, 10(1):83–98, 1988.

[19] Henry Helson. *Harmonic analysis*. Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1983.

[20] S.J. Hogan. Some effects of surface tension on steep water waves. Part 2. *Journal of Fluid Mechanics*, 96:417–445, 1 1980.

[21] S.J. Hogan. The superharmonic normal mode instabilities of nonlinear deep-water capillary waves. *J. Fluid Mech.*, 190:165–177, 1988.

[22] T.Y. Hou, J.S. Lowengrub, and M.J. Shelley. Removing the stiffness from interfacial flows with surface tension. *J. Comput. Phys.*, 114(2):312–338, 1994.

[23] T.Y. Hou, J.S. Lowengrub, and M.J. Shelley. The long-time motion of vortex sheets with surface tension. *Phys. Fluids*, 9(7):1933–1954, 1997.

[24] John K. Hunter and Bruno Nachtergaele. *Applied analysis*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[25] Vera Mikyoung Hur. No solitary waves exist on 2d deep water. *Nonlinearity*, 25(12):3301, 2012.
[26] W. Kinnersley. Exact large amplitude capillary waves on sheets of fluid. *Journal of Fluid Mechanics*, 77:229–241, 8 1976.
[27] B. Kinsman. *Wind Waves: Their Generation and Propagation on the Ocean Surface*. Dover Publications, 2012.
[28] T. Levi-Civita. Determinazione rigorosa delle onde irrotazionali periodiche in acqua profonda. *Rend. Accad. Lincei.*, 33:141–150, 1924.
[29] M.S. Longuet-Higgins. Limiting forms for capillary-gravity waves. *Journal of Fluid Mechanics*, 194:351–375, 8 1988.
[30] L.F. McGoldrick. On Wilton's ripples: a special case of resonant interactions. *J. Fluid Mech*, 42(1):193–200, 1970.
[31] D.I. Meiron and P.G. Saffman. Overhanging interfacial gravity waves of large amplitude. *J. Fluid. Mech.*, 129:213–218, 1983.
[32] PAUL A. MILEWSKI, J.-M. VANDEN-BROECK, and ZHAN WANG. Dynamics of steep two-dimensional gravity?capillary solitary waves. *Journal of Fluid Mechanics*, 664:466–477, 12 2010.
[33] David P Nicholls and Fernando Reitich. Stable, high-order computation of traveling water waves in three dimensions. *European Journal of Mechanics-B/Fluids*, 25(4):406–424, 2006.
[34] H. Okamoto. Uniqueness of Crapper's pure capillary waves of permanent shape. *J. Math. Sci. Univ. Tokyo*, 12(1):67–75, 2005.
[35] H. Okamoto and M. Shōji. *The mathematical theory of permanent progressive water-waves*, volume 20 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.
[36] AJ Roberts. Highly nonlinear short-crested water waves. *Journal of Fluid Mechanics*, 135:301–21, 1983.
[37] AJ Roberts and DH Peregrine. Notes on long-crested water waves. *Journal of Fluid Mechanics*, 135:323–35, 1983.
[38] AJ Roberts and LW Schwartz. The calculation of nonlinear short-crested gravity waves. *Physics of Fluids*, 26:2388, 1983.
[39] P.G. Saffman and H.C. Yuen. Finite-amplitude interfacial waves in the presence of a current. *J. Fluid Mech.*, 123:459–476, 1982.
[40] Leonard W Schwartz. Computer extension and analytic continuation of Stokes' expansion for gravity waves. *Journal of Fluid Mechanics*, 62(3):553–578, 1974.
[41] Leonard W Schwartz and Jean-Marc Vanden-Broeck. Numerical solution of the exact equations for capillary–gravity waves. *Journal of Fluid Mechanics*, 95(01):119–139, 1979.
[42] G. Stokes. On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.*, 8:441–455, 1847.
[43] S. Tanveer. Some analytical properties of solutions to a two-dimensional steadily translating inviscid bubble. *Proc. Roy. Soc. London Ser. A*, 452(1949):1397–1410, 1996.
[44] R. Tiron and W. Choi. Linear stability of finite-amplitude capillary waves on water of infinite depth. *Journal of Fluid Mechanics*, 696:402–422, 3 2012.
[45] J. F. Toland. On a pseudo-differential equation for Stokes waves. *Arch. Ration. Mech. Anal.*, 162(2):179–189, 2002.
[46] R.E.L. Turner and J.-M. Vanden-Broeck. The limiting configuration of interfacial gravity waves. *Phys. Fluids*, 29:372–375, 1986.
[47] J.-M. Vanden-Broeck. Wilton ripples generated by a moving pressure distribution. *Journal of Fluid Mechanics*, 451(1):193–201, 2002.
[48] J.-M. Vanden-Broeck and F. Dias. Gravity-capillary solitary waves in water of infinite depth and related free surface flows. *J. Fluid Mech*, 240:549–557, 1992.
[49] J.-M. Vanden-Broeck and J.B. Keller. A new family of capillary waves. *Journal of Fluid Mechanics*, 98:161–169, 4 1980.
[50] Z. Wang. Personal Communication, 2013.
[51] J.R. Wilton. LXXII. On ripples. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 29(173):688–700, 1915.